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1. Introduction.

Poisson's integral equation

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\, \phi(t)\, dt}{(t-x)^2 + a^2} \qquad \begin{array}{c} -\infty < x < \infty \\ a > 0 \end{array} \tag{1.1}$$

may be regarded as the one which arises in connection with the problem of finding a potential function $\psi(x,y)$ which satisfies

$$\psi_{xx} + \psi_{yy} = 0 \qquad y > 0$$

and which is equal to $f(x)$ along the line $y = a > 0$. If $\psi(x,0+) = \phi(x)$; and if $\psi(x,y)$ is regular at infinity then

$$\psi(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y\, \phi(t)\, dt}{(t-x)^2 + y^2} .$$

If $\psi(x,a) = f(x)$ the function $\phi(t)$ must satisfy Poisson's equation. The function $f(x)$ is called the Poisson transform of $\phi(t)$ and we will write

$$f(x) = P_a \phi \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\, \phi(t)\, dt}{(t-x)^2 + a^2} .$$

We can say that Poisson's integral equation poses the problem of finding the boundary value of a function known to be harmonic in a half-plane when the boundary is inaccessible, but measurements can be taken along a line parallel to the boundary. This kind of problem arises in various physical investigations. For example, the solution of the problem is of considerable value for the interpretations of observations derived from certain methods of geological prospecting which depend on a survey of a potential field such as the gravimetric or magnetic field of the

earth. In prospecting, the problem is to locate and evaluate concentrations of matter which cause anomalous fields and hence the quantity a may also be unknown at the start.

Snow's formula for the solution of (1.1) is well known; but not long ago, Pollard [1] showed that the Poisson transform obeys the law

$$P_a P_b \phi = P_{a+b} \phi \tag{1.2}$$

and he used this to establish Snow's inverstion formula under more general assumptions than those used previously. The purpose of this report is to present other proofs, both new and old, of Pollard's result; and to show some extensions and applications of the property (1.2).

Section 2 contains an exposition and simplification of Pollard's proof besides some other methods for the solution of (1.1). In Section 3 it is shown that if

$$P_\lambda \equiv \frac{(1-\lambda^2)}{2\pi} \int_0^{2\pi} \frac{\phi(\alpha)\, d\alpha}{1 + \lambda^2 - 2\lambda \cos(\alpha-\theta)}$$

then

$$P_{\lambda_1} P_{\lambda_2} \phi = P_{\lambda_1 \lambda_2} \phi$$

and this is used to solve Poisson's equation for the circle, namely

$$f(\theta) = P_\lambda \phi .$$

Section 4 is concerned with a generalization of Poisson's equation, the integral equation which arises when we wish to determine the boundary value of a two-dimensional harmonic functions from its

behavior along an interior closed path. The results of Section 3 are used to solve a certain biharmonic problem posed in Section 5. In Section 6 various inversion formulas derived in Section 2 are examined from the practical point of view. Several approximation procedures are suggested and Section 6 includes a derivation of an iteration procedure for the solution of Poisson's equation. Finally, Section 7 shows how some of the foregoing results can be extended to analogous potential problems in higher dimensions.

2. Some Methods for the Solution of Poisson's Integral Equation.

We assume that $\phi(\xi)$ is integrable on each finite interval, and that each of

$$(2.1) \qquad \int_{c}^{\infty} \frac{\phi(t)\,dt}{t^2} \; ; \qquad \int_{-\infty}^{-c} \frac{\phi(t)\,dt}{t^2} \qquad c > 0$$

exists. Under this assumption, the integral

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a\,\phi(t)\,dt}{(t-x)^2+a^2} = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \left[\frac{1}{t-x-ia} - \frac{1}{t-x+ia}\right] \phi(t)\,dt$$

exists, if $a > 0$; and it represents a function of x which is analytic when x is real. Hence a solution of Poisson's integral equation

$$(2.2) \qquad f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a\,\phi(t)\,dt}{(t-x)^2+a^2} \qquad \begin{array}{c} -\infty < x < \infty \\ a > 0 \end{array}$$

cannot exist if the prescribed f(x) fails to be differentiable an infinite number of times at any value of x which is real.

I. Introduce

$$F(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\, \phi(t)\, dt}{(t-z)^2+a^2} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} [\frac{1}{t-z-ia} - \frac{1}{t-z+ia}]\, \phi(t)\, dt$$

where $z = x+iy$. This function of the complex variable z is analytic everywhere except along the lines $z = t-ia$ and $z = t+ia$, each of which is parallel to the real axis and a units distant from it. Since $F(z)$ coincides with $f(x)$ when z is real, the integral which defines $F(z)$ gives the analytic continuation of $f(x)$ into the strip $-a < \mathrm{Im}\, z = y < a$. In other words, if $\phi(t)$ is to exist, the prescribed $f(x)$ must be analytic on the real axis and it must allow an analytic continuation of itself into the strip $-a < \mathrm{Im}\, z < a$. If $f(x)$ allows such a continuation, the continuation is unique.

We proceed under the assumption that $f(x)$ satisfies the analyticity conditions which are required for the existence of $\phi(t)$. If $\varepsilon > 0$ we have

$$\lim_{\varepsilon \to 0} \{f[x+i(a-\varepsilon]+f[x-i(a-\varepsilon)]\}$$
$$= \lim_{\varepsilon \to 0} \{\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon\, \phi(t)\, dt}{(t-x)^2+\varepsilon^2} + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(2a-\varepsilon)\, \phi(t)\, dt}{(t-x)^2+(2a-\varepsilon)^2}\} .$$

It is known that for almost all values of x

$$\phi(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon\, \phi(t)\, dt}{(t-x)^2+\varepsilon^2} .$$

Hence for almost all values of x

$$\phi(x) = \lim_{\varepsilon \to 0} \{f[x+i(a-\varepsilon)] + f[x-i(a-\varepsilon)]\} - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2a\, \phi(t)\, dt}{(t-x)^2+4a^2} .$$

The integral in the last equation is equal to $\psi(x,2a)$ where $\psi(x,y)$ is the potential function

$$\psi(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y\, \phi(t)\, dt}{(t-x)^2+y^2} \qquad y > 0.$$

According to (2.2), this potential function at $y = a$, namely $\psi(x,a)$, is prescribed to be equal to the analytic function $f(x)$. Consequently, the Poisson integral formula for a function harmonic in a half-plane shows that $\psi(x,y)$ is given by

$$\psi(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(y-a)\, f(t)\, dt}{(t-x)^2 + (y-a)^2} \tag{2.3}$$

when $y > a$. The analyticity condition on $f(x)$ insures the uniqueness of $\psi(x,y)$. It follows from (2.3) that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2a\, \phi(t)\, dt}{(t-x)^2+4a^2} = \psi(x,2a) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\, f(t)\, dt}{(t-x)^2+a^2}\ .$$

We therefore conclude that for almost all x

$$\phi(x) = \lim_{\varepsilon \to 0} \{f[x+i(a-\varepsilon)] + f[x-i(a-\varepsilon)]\} - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\, f(t)\, dt}{(t-x)^2+a^2} \tag{2.4}$$

is the solution of Poisson's integral equation

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\, \phi(t)\, dt}{(t-x)^2+a^2} \qquad \begin{array}{c} -\infty < x < \infty \\ a > 0 \end{array}$$

in which $f(x)$ and $\phi(x)$ satisfy the conditions stated above. The formula (2.4) can be expressed in a slightly more compact and theoretically more useful form if we use the fact that $f(x)$ must allow an analytic extension of itself into the strip $-a < \quad z < a$. If we start with

$$\phi(x) = \lim_{\varepsilon \to 0} \{f[x+i(a-2\varepsilon)] + f[x-i(a-2\varepsilon)]\} - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\, f(t)\, dt}{(t-x)^2+a^2}$$

on what is the same thing,

$$\phi(x) = \lim_{\varepsilon \to 0}\ 2\, \mathrm{Im}\, i\ f[x+i(a-2\varepsilon)] - \frac{1}{\pi}\ \mathrm{Im} \int_{-\infty}^{\infty} \frac{f(t)\, dt}{t-x-ia}\ ,$$

an application of the Cauchy integral formula gives

$$\phi(x) = \lim_{\varepsilon \to 0} \{\frac{1}{\pi} \operatorname{Im} \int_{-\infty}^{\infty} \frac{f(t)\,dt}{t-x-i(a-2\varepsilon)} - \frac{1}{\pi} \operatorname{Im} \int_{-\infty+i(a-\varepsilon)}^{\infty+i(a-\varepsilon)} \frac{f(t)\,dt}{t-x-i(a-2\varepsilon)}\}$$

$$- \frac{1}{\pi} \operatorname{Im} \int_{-\infty}^{\infty} \frac{f(t)\,dt}{t-x-ia}$$

$$\phi(x) = -\lim_{\varepsilon \to 0} \frac{1}{\pi} \operatorname{Im} \int_{-\infty+i(a-\varepsilon)}^{\infty+i(a-\varepsilon)} \frac{f(t)\,dt}{t-x-i(a-2\varepsilon)} .$$

A change in the variable of integration namely, $t = z + i(a-\varepsilon)$, yields

(2.5) $$\phi(x) = -\lim_{\varepsilon \to 0} \operatorname{Im} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f[z+i(a-\varepsilon)]\,dz}{z - x + i\varepsilon}$$

or its equivalent

(2.6) $$\phi(x) = \lim_{\varepsilon \to 0} \operatorname{Im} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f[z-i(a-\varepsilon)]\,dz}{z - x - i\varepsilon} .$$

II. H. Pollard [1] deduced the inverstion formula (2.4) for the Poisson transform

(2.7) $$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\,\phi(t)\,dt}{(t-x)^2+a^2} \equiv P_a\,\phi$$

after establishing the interesting fact that this transform obeys the semi-group property

(2.8) $$P_b\,P_a\,\phi = P_{a+b}\,\phi .$$

After the proof of (2.8) Pollard's method of analysis is given by the following sequence of equations:

$$P_y f = P_y P_a \phi = P_{y+a}\phi = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(y+a)\ \phi(t)\ dt}{(t-x)^2+(y+a)^2}$$

$$f(s+iy) + f(x-iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \left[\frac{a}{(t-x-iy)^2+a^2} + \frac{a}{(t-x+iy)^2+a^2}\right]\phi(t)\ dt$$

$$f(x+iy) + f(x-iy) - P_y f$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \left[\begin{array}{l} \dfrac{1}{t-x-i(y+a)} - \dfrac{1}{t-x-i(y-a)} \\ + \dfrac{1}{t-x+i(y-a)} - \dfrac{1}{t-x+i(y+a)} \\ - \dfrac{1}{t-x-i(y+a)} + \dfrac{1}{t-x+i(y+a)} \end{array}\right] \phi(t)\ dt$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(a-y)\ \phi(t)\ dt}{(t-x)^2+(a-y)^2}$$

$$\lim_{y \to a-} [f(x+iy)+f(x-iy)] - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)\ dt}{(t-x)^2+a^2} = \phi(x)\ .$$

Pollard's proof of the semi-group property (2.8) does not depend on the use of harmonic functions. The Poisson transform, however, is most likely to arise in connection with problems in potential theory, and a proof based on this theory may be desirable. We proceed to give a proof of this kind.

If

$$f(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a\ \phi(\xi)\ d\xi}{(\xi-t)^2+a^2} \equiv P_a\ \phi$$

then

$$P_b P_a \phi = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{b}{(t-x)^2+b^2}\int_{-\infty}^{\infty} \frac{a\ \phi(\xi)\ d\xi}{(\xi-t)^2+a^2}\ dt\ .$$

Now the function

$$\omega(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(t-x)^2+y^2} \cdot \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a\ \phi(\xi)\ d\xi}{(\xi-t)^2+a^2}\, dt$$

is a potential function for $y > 0$ and

$$P_b\ P_a\ \phi = \omega(x,b)\ .$$

We also see that

$$\omega(x,0+) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a\ \phi(\xi)\ d\xi}{(\xi-x)^2+a^2}$$

$$= \lim_{y\to 0} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(y+a)\ \phi(\xi)\ d\xi}{(\xi-x)^2+(y+a)^2}\ .$$

Furthermore, the function

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(y+a)\ \phi(\xi)\ d\xi}{(\xi-x)^2+(y+a)^2}$$

is harmonic in the upper half-plane, and since it is equal to $\omega(x,0+)$ when $y \to 0$, it follows from the appropriate uniqueness theorem that

$$\omega(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(y+a)\ \phi(\xi)\ d\xi}{(\xi-x)^2+(y+a)^2}\ .$$

We conclude from this that

$$P_b\ P_a\ \phi = \omega(x,b) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(b+a)\ \phi(\xi)\ d\xi}{(\xi-x)^2+(b+a)^2} \equiv P_{a+b}\phi.$$

The idea of this proof can easily be extended so that it can be applied in the analysis of certain integral equations which may be regarded as generalizations of Poisson's integral equation. Some of these equations are considered in the sequel.

We have seen that the semi-group property implies the inversion formula. We remark in passing that conversely, it is not difficult to prove that the semi-group property is implied by the inverseion formula.

III. The Poisson transform of ϕ is a convolution integral and this means that we can also invert

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\,\phi(t)\,dt}{(t-x)^2+a^2}$$

by using Fourier transforms. Let $\phi(t)$ be a function such that we can assert that the Fourier transform of the integral is equal to the product of the transforms

$$\Phi(w) = \int_{-\infty}^{\infty} e^{iwx}\,\phi(x)\,dx$$

and

$$e^{-a|w|} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{iwx}\,a\,dx}{x^2+a^2} .$$

We then have

$$F(w) = e^{-a|w|}\,\Phi(w)$$

where

$$F(w) = \int_{-\infty}^{\infty} e^{ixw}\,f(x)\,dx .$$

It follows that the transform of ϕ is

$$\begin{aligned} \Phi(w) &= e^{a|w|}\,F(w) \\ &= [e^{a|w|} + e^{-a|w|} - e^{-a|w|}]\,F(w) \\ &= [e^{aw} + e^{-aw} - e^{-a|w|}]\,F(w) \end{aligned}$$

and hence by inversion

$$\phi(x) = f(x+ia) + f(x-ia) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\,f(t)\,dt}{(t-x)^2+a^2} . \tag{2.9}$$

This proof is formally simple and elegant; but the justification of the formal procedure involves a much more complicated analysis in which $\phi(x)$ must apparently be subjected to severer restrictions than the mild ones stipulated in the previous methods. For example, the above theory at least requires the existence of the transform $\Phi(w)$, but the non-existence of $\Phi(w)$ does not interfere with the proofs I and II.

The formula (2.9) is often called Snow's formula. C. Snow [2] derived it in 1924 by using operational procedures based on ordinary Fourier transform theory, and subject to rather severe restrictions.

The confining conditions imposed by the ordinary Fourier transform theory can of course be considerably lightened if we think of this theory in terms of generalized functions and their transforms. The use of the more sophisticated theory also allows us to retain formal simplicity and elegance; but it should be remarked that the theory of generalized functions introduces an apparatus which is much more advanced than the methods of proof presented here, in this section. Nevertheless, the results of an appeal to the theory of generalized functions are worth noting. According to this theory the inverse of

$$\Phi(w) = e^{a|w|} F(w)$$

is

$$\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixw} e^{a|w|} \int_{-\infty}^{\infty} e^{itw} f(t)\, dt\, dw$$

or

$$(2.10) \qquad \phi(x) = \frac{1}{\pi} \operatorname{Re} \int_0^\infty e^{aw} e^{-ixw} \int_{-\infty}^\infty e^{itw} f(t)\, dt\, dw$$

where the transforms are to be interpreted as generalized transforms. Thus, if we write

$$f(x) = P_a \phi \equiv \frac{1}{\pi} \int_{-\infty}^\infty \frac{a\, \phi(t)\, dt}{(t-x)^2+a^2}$$

and symbolically

$$\phi = \frac{1}{P_a} f ,$$

we can interpret $[P_a]^{-1}$ to mean

$$(2.11) \qquad \frac{1}{P_a} f \equiv \frac{1}{\pi} \operatorname{Re} \int_0^\infty e^{aw} e^{-ixw} \int_{-\infty}^\infty e^{itw} f(t)\, dt\, dw$$

where we assume $a \geq 0$. From the present point of view, the operator $P_a f$ is defined by

$$(2.12) \qquad P_a f \equiv \frac{1}{\pi} \int_{-\infty}^\infty \frac{a\, f(t)\, dt}{(t-x)^2+a^2}$$

$$\equiv \frac{1}{\pi} \operatorname{Re} \int_0^\infty e^{-aw} e^{-ixw} \int_{-\infty}^\infty e^{itw} f(t)\, dt\, dw .$$

IV. H. Bateman [3] showed that the inversion formula for the Poisson transform can be found by using the theory of continuation of potential functions. The following is a somewhat amplified account of his method.

If a solution of

$$f(x) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{a\, \phi(t)\, dt}{(t-x)^2+a^2}$$

exists, then $f(z)$ must be analytic for $-a < \quad z < a$. The real part

of f(z), namely, Re f(x+iy) is harmonic in $-a < y < a$ and it reduces to f(x) when $y = 0$. As we know, Poisson's equation may be regarded as prescribing the value of the potential function

$$\psi(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y\,\phi(t)\,dt}{(t-x)^2+y^2} \qquad y > 0$$

at $y = a$, that is $\psi(x,a) = f(x)$. Now $\psi(x,y)$ is the real part of a function $F(x+iy)$ which is analytic in $z = x+iy$ for $y > 0$. In effect, the integral equation poses the problem of deducing

$$\lim_{\varepsilon \to 0} \operatorname{Re} F(x+i\varepsilon)$$

from

$$\operatorname{Re} F(x+ia) = f(x) .$$

The function $\operatorname{Re} F(x+iy+ia)$ is harmonic in the strip $-a < y < a$. The difference

$$\operatorname{Re} \{F(x+iy+ia) - f(x+iy)\}$$

is harmonic in $-a < y < a$ and it <u>vanishes</u> for $y = 0$. Therefore, by harmonic continuation from the strip $0 < y < a$ into the strip $-a < y < 0$ we have

$$- \operatorname{Re} \{F(x+iy+ia)-f(x+iy)\} = - \quad \{F(x+iy+ia)-f(x+iy)\}$$

or

$$\operatorname{Re} \{F(x-iy+ia)+F(x+iy+ia)\} = f(x+iy)+f(x-iy) .$$

If we let $y \to a-$ we find

$$\lim_{y\to a-} \operatorname{Re} F(x-iy+ia) = \psi(x,0+) = \phi(x)$$

$$= \lim_{y\to a-} \{f(x+iy)+f(x-iy)\} - \operatorname{Re} F(x+2ia) .$$

For $y > a$

$$\operatorname{Re} F(x+iy) = \psi(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(y-a)\ f(t)\ dt}{(t-x)^2+(y-a)^2}$$

and from this

$$\operatorname{Re} F(x+2ia) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\ f(t)\ dt}{(t-x)^2+a^2} .$$

Thus once again, we see that

$$\phi(x) = \lim_{y\to a-} \{f(x+iy) + f(x-iy)\} - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\ f(t)\ dt}{(t-x)^2+a^2} .$$

At first sight it may seem that we can start with

$$\operatorname{Re} F(x+ia) = f(x)$$

and then immediately find $\phi(x)$ from

$$\phi(x) = \lim_{\varepsilon\to 0} \operatorname{Re} F(x+i\varepsilon) = \lim_{\varepsilon\to 0} f(x-ia+i\varepsilon) .$$

However, this is only effective if we can see how to express $f(x)$ in the form

$$f(x) = \operatorname{Re} F(x+iy)\Big|_{y=a}$$

where $F(x+iy)$ is <u>analytic in</u> $z = x+iy$ <u>for</u> $y > 0$. [This, in reality, is the essence of the problem of solving the integral equation; it is what the inversion formula accomplishes.] For example, if

$$f(x) = \sin x = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\ \phi(t)\ dt}{(t-x)^2+a^2}$$

then the solution $\phi(t)$ is not

$$\lim_{\varepsilon \to 0} \sin\ (x-ia+i\varepsilon) = \sin\ (x-ia)$$

because this is not real and the inversion formula shows it must be when $f(x)$ is real. On the other hand, if we recognize that

$$f(x) = \sin x = -\text{Re}\ i\ e^a\ e^{ix-a} = -\ e^a\ \text{Re}\ i\ e^{i(x+iy)}\Big|_{y=a}$$

then

$$f(x-ia) = -e^a\ \text{Re}\ i\ e^{i(x-ia+iy)}\Big|_{y=a} = -e^a\ \text{Re}\ i\ e^{ix} = e^a\ \sin x$$

does in fact give the solution.

In connection with the remarks in the last paragraph it should be noted that the expression of $f(x)$ in the form

$$f(x) = \text{Re}\ F(x+iy)\Big|_{y=a}$$

is immediate if we use the theory of generalized functions. This theory gives

$$f(x) = \frac{1}{\pi}\ \text{Re} \int_0^{\infty} e^{i\xi x} \int_{-\infty}^{\infty} e^{-i\xi t}\ f(t)\ dt\ d\xi$$

$$= \frac{1}{\pi}\ \text{Re} \int_0^{\infty} e^{i\xi(x+iy)}\ e^{a\xi} \int_{-\infty}^{\infty} e^{-i\xi t}\ f(t)\ dt\ d\xi\Big|_{y=a+}$$

provided the generalized transforms exist. We suppose that $f(z)$ is analytic in $-a < z < a$ and such that

$$F(x+iy) = \frac{1}{\pi} \int_0^{\infty} e^{i\xi(x+iy)}\ e^{a\xi} \int_{-\infty}^{\infty} e^{-i\xi t}\ f(t)\ dt\ d\xi$$

is analytic for $y > 0$. Then

$$\phi(x) = f(x-ia) = \frac{1}{\pi} \operatorname{Re} \int_0^\infty e^{a\xi} \; e^{-i\xi x} \int_{-\infty}^{\infty} e^{i\xi t} \; f(t) \; dt \; d\xi$$

which is the formula (2.10). If $f(t) = \sin bt$, the generalized transform is

$$\int_{-\infty}^{\infty} e^{i\xi t} \sin bt \; dt = i\pi[\delta(\xi-b) - \delta(\xi+b)]$$

where $\delta(\tau-\tau_1)$ is the generalized function defined by

$$\int_{-\infty}^{\infty} \delta(\tau-\tau_1) \; A(\tau) \; d\tau = A(\tau_1) \; .$$

Hence

$$\phi(x) = f(x-ia) = \operatorname{Re} i \int_0^\infty e^{a\xi} \; e^{-i\xi x}[\delta(\xi-b)-\delta(\xi+b)] \; d\xi$$

$$= \operatorname{Re} i \; e^{ab} \; e^{-ibx} = e^{ab} \sin bx \; .$$

The above methods suggest that we can extend our considerations to geometric configurations other than the half-plane. They also suggest that problems similar to the one discussed above can be solved for other elliptic partial differential equations besides the two-dimensional potential equation.

3. Poisson's Integral Equation for the Circle.

The equation

$$(3.1) \qquad f(\theta) = \frac{r^2-a^2}{2\pi} \int_0^{2\pi} \frac{\phi(\alpha)\, d\alpha}{r^2+a^2-2ar\cos(\theta-\alpha)} \qquad \begin{matrix} a < r \\ 0 \leq \theta \leq 2\pi \end{matrix}$$

can be referred to as Poisson's integral equation for the circle. It can be identified as the one which needs to be solved in connection with the problem of finding a potential function $\psi(\rho,\theta)$ which satisfies $\rho^2\psi_{\rho\rho} + \rho\psi_{\psi} + \psi_{\theta\theta} = 0$ for $\rho < r$ and which is equal to $f(\theta)$ along the circle $\rho = a < r$. We assume that $f(\alpha)$ is merely integrable. If $\psi(r-,\theta) = \phi(\theta)$ almost everywhere we have

$$(3.2) \qquad \psi(\rho,\theta) = \frac{r^2-\rho^2}{2\pi} \int_0^{2\pi} \frac{\phi(\alpha)\, d\alpha}{r^2+\rho^2-2\rho r\cos(\theta-\alpha)}$$

and if $\psi(a,\theta) = f(\theta)$ then $\phi(\alpha)$ must satisfy (3.1).

The integral in (3.1) can be written

$$\frac{r^2-a^2}{2\pi} \int_0^{2\pi} \frac{\phi(\alpha)\, d\alpha}{[a-re^{-i(\alpha-\theta)}][a-re^{i(\alpha-\theta)}]}$$

or, if we replace $e^{i\theta}$ by z, as

$$\frac{r^2-a^2}{2\pi} \int_0^{2\pi} \frac{\phi(\alpha)\, d\alpha}{[a-re^{-i\alpha}z][a-\frac{re^{i\alpha}}{z}]} .$$

This integral defines a function of the complex variable $z = \rho e^{i\theta}$, analytic in the domain $a/r < |z| < r/a$. It shows that (3.1) possesses a solution only if the prescribed function $f(\theta)$ is an analytic periodic function of θ which allows continuation into the annular domain defined above. If this condition is satisfied, the equation (3.1) can presumably be solved by using any

using any one of the methods of Section 2. [In order to solve (3.1) by the transform method III, this method must of course be changed to a finite transform method.] Let us choose method II which depends on the existence of a semi-group property of the transform because of the four methods this one seems the most challenging, and most likely to involve a new idea.

If we put $\lambda = a/r < 1$ we can write (3.1) in the form

$$f(\theta) = \frac{1-\lambda^2}{2\pi} \int_0^{2\pi} \frac{\phi(\alpha)\, d\alpha}{1+\lambda^2-2\lambda\cos(\alpha-\theta)} \equiv P_\lambda \phi \,. \tag{3.3}$$

With $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$, the repeated transform $P_{\lambda_1}P_{\lambda_2}\phi$ is

$$P_{\lambda_1}P_{\lambda_2}\phi = \frac{(1-\lambda_1^2)(1-\lambda_2^2)}{4\pi^2} \int_0^{2\pi} \frac{1}{1+\lambda_1^2-2\lambda_1\cos(\beta-\theta)} \int_0^{2\pi} \frac{\phi(\alpha)\, d\alpha\, d\beta}{1+\lambda_2^2-2\lambda_2\cos(\alpha-\beta)} .$$

The function

$$\omega(\rho,\theta) = \frac{(1-\rho^2)}{2\pi} \int_0^{2\pi} \frac{1}{1+\rho^2-2\rho\cos(\beta-\theta)} \cdot \left[\frac{(1-\lambda_2^2)}{2\pi} \int_0^{2\pi} \frac{\phi(\alpha)\, d\alpha}{1+\lambda_2^2-2\lambda_2\cos(\alpha-\beta)}\right] d\beta$$

is a potential function for $\rho < 1$ and

$$P_{\lambda_1}P_{\lambda_2}\phi = \omega(\lambda_1,\theta) \,.$$

We also see that

$$\omega(1-,\theta) = \frac{1-\lambda_2^2}{2\pi} \int_0^{2\pi} \frac{\phi(\alpha)\, d\alpha}{1+\lambda_2^2-2\lambda_2\cos(\alpha-\theta)}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \phi(\alpha)\, \mathrm{Re}\, \left\{\frac{1+\lambda_2 e^{i(\theta-\alpha)}}{1-\lambda_2 e^{i(\theta-\alpha)}}\right\} d\alpha$$

$$= \lim_{\rho\to 1} \frac{1}{2\pi} \int_0^{2\pi} \phi(\alpha) \text{ Re } \{\frac{1+\lambda_2\rho \ e^{i(\theta-\alpha)}}{1+\lambda_2\rho \ e^{i(\theta-\alpha)}}\} \, d\alpha$$

$$= \lim_{\rho\to 1} \frac{1}{2\pi} \int_0^{2\pi} \phi(\alpha) \text{ Re } \{\frac{1+z\lambda_2 \ e^{-i\alpha}}{1-z\lambda_2 \ e^{-i\alpha}}\} \, d\alpha$$

where $z = \rho \ e^{i\theta}$. Since

$$\frac{1}{2\pi} \int_0^{2\pi} \phi(\alpha) \text{ Re } \{\frac{1+z\lambda_2 \ e^{-i\alpha}}{1-z\lambda_2 e^{-i\alpha}}\} \, d\alpha$$

is a potential function for $|z| < 1$, and since it is equal to $\omega(1-,\theta)$ when $|z| \to 1$ it follows that

$$\omega(\rho,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\alpha) \text{ Re } \{\frac{1+z\lambda_2 e^{-i\alpha}}{1-z\lambda_2 e^{-i\alpha}}\} \, d\alpha \ .$$

We can now see that

$$P_{\lambda_1} P_{\lambda_2} \phi = \omega(\lambda_1,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\alpha) \text{ Re } \{\frac{1+\lambda_1\lambda_2 e^{i(\theta-\alpha)}}{1-\lambda_1\lambda_2 e^{i(\theta-\alpha)}}\} \, d\alpha$$

$$= \frac{[1-(\lambda_1\lambda_2)^2]}{2\pi} \int_0^{2\pi} \frac{\phi(\alpha) \, d\alpha}{1+(\lambda_1\lambda_2)^2-2\lambda_1\lambda_2\cos(\alpha-\theta)}$$

and hence that $P_\lambda\phi$ obeys the semi-group property

$$P_{\lambda_1} P_{\lambda_2} \phi = P_{\lambda_1\lambda_2} \phi \ .$$

This property can be used to invert (3.1) in the following way. The equation (3.1) can be written

$$f(\theta) = P_\lambda \phi \equiv \frac{1}{2\pi} \int_0^{2\pi} [\frac{e^{i\alpha}}{e^{i\alpha}-\lambda e^{i\theta}} + \frac{\lambda \ e^{-i\theta}}{e^{-i\alpha}-\lambda e^{-i\theta}}] \, \phi(\alpha) \, d\alpha \ .$$

Then

$$f(\theta+i\ \ln\rho) + f(\theta-i\ \ln\rho) - P_\rho f = f(\theta+i\ \ln\rho)+f(\theta-i\ \ln\rho)-P_{\rho\lambda}\phi$$

$$f(\theta+i\ \ln\rho) + f(\theta-i\ \ln\rho) - P_\rho f$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\begin{array}{cc} \dfrac{\rho e^{i\alpha}}{\rho e^{i\alpha}-\lambda e^{i\theta}} & + \dfrac{\lambda\rho e^{-i\theta}}{e^{-i\alpha}-\lambda\rho e^{-i\theta}} \\ + \dfrac{e^{i\alpha}}{e^{i\alpha}-\lambda\rho e^{i\theta}} & + \dfrac{\lambda\ e^{-i\theta}}{\rho e^{-i\alpha}-\lambda e^{-i\theta}} \\ - \dfrac{e^{i\alpha}}{e^{i\alpha}-\lambda\rho e^{i\theta}} & - \dfrac{\lambda\rho e^{-i\theta}}{e^{-i\alpha}-\lambda\rho e^{-i\theta}} \end{array}\right]\phi(\alpha)\ d\alpha$$

from which

$$f(\theta+i\ \ln\rho) + f(\theta-i\ \ln\rho) - \frac{(1-\rho^2)}{2\pi}\int_0^{2\pi}\frac{f(\alpha)\ d\alpha}{1+\rho^2-2\rho\ \cos(\alpha-\theta)}$$

$$= \frac{(\rho^2-\lambda^2)}{2\pi}\int_0^{2\pi}\frac{\phi(\alpha)\ d\alpha}{\rho^2+\lambda^2-2\rho\lambda\ \cos(\alpha-\theta)} .$$

Finally, if we let $\rho \to \lambda+$ we find

(3.4)
$$\phi(\theta) = \lim_{\rho\to\lambda+}\{f(\theta+i\ \ln\rho)+f(\theta-i\ \ln\rho)\} - \frac{(1-\lambda^2)}{2\pi}\int_0^{2\pi}\frac{f(\alpha)\ d\alpha}{1+\lambda^2-2\lambda\ \cos(\alpha-\theta)}$$

for almost all values of θ. This is the inversion formula for (3.3).

4. Some Generalizations of Poisson's Integral Equation.

Poisson's integral equation suggests a more general problem. Let D be a simply connected domain whose piecewise smooth boundary C, is given by

$$z = z_1(t) \qquad a \leq t < b$$

and let C_2, given by

$$z = z_2(\tau) \qquad c \leq \tau < d\ ,$$

be a simple piecewise smooth closed path in the interior of D. Let $\psi(x,y)$ be harmonic in D. Suppose that at the boundary C_1 the limit of $\psi(x,y)$ is $\phi(t)$ and that if (x,y) is a point in D then $\psi(x,y)$ is given by

$$\psi(x,y) = \int_a^b \bar{G}(x,y;t)\ \phi(t)\ dt\ .$$

The problem we have in mind is this. Solve the integral equation

$$(4.1)\quad f(\tau) \equiv \mathrm{Re}\{H[z_2(\tau)]\} = \int_a^b \bar{G}[x_2(\tau),y_2(\tau);t]\ \phi(t)\ dt \qquad c \leq \tau < d$$

where $z_2(\tau) = x_2(\tau) + iy_2(\tau)$ is on C_2. We assume that $H(z)$ is analytic for z on C_2 and for z in the domain bounded by C_2 and C_1. We also assume of course that the prescribed function $f(\tau) \equiv \mathrm{Re}\{H[z_2(\tau)]\}$ possesses all of the properties implied by the integral representation on the right hand side of the equation.

If we set

$$\psi(x,y) = \mathrm{Re}\{F(z)\}$$

where z is an analytic function of $z = x+iy$ for z in D, then the solution of the integral equation (4.1) requires the deduction of

$$\lim_{z \to z_1(t)} \operatorname{Re}\{F(z)\} = \phi(t)$$

from

$$\operatorname{Re}\{F[z_2(\tau)]\} = f(\tau) = \operatorname{Re}\{H[z_2(\tau)]\} \ .$$

This can be accomplished with the aid of conformal mapping.

Let $\zeta = m(z)$ be the function which maps the domain enclosed by C_2 into the upper half $\bar{U}$ of a ζ-plane. Under this mapping the image of C_1 will be C^*, in general a curve not in $\bar{U}$, but in the Riemann surface formed by properly connecting superimposed ζ-plane sheets in accordance with $\zeta = m(z)$. However, we assume here for simplicity that C^* is in the lower half-plane on the same sheet as $\bar{U}$. The image of D is then the domain D^* which contains the point at infinity in the ζ-plane and which has C^* for its boundary.

Let the inverse of

$$\zeta = m(z)$$

be

$$z = M(\zeta) \ .$$

With this we have

$$\operatorname{Re}\{F[M(\xi)]\} = \operatorname{Re}\{H[M(\xi)]\} \ .$$

Now the difference

$$\operatorname{Re}\{F[M(\zeta)] - H[M(\zeta)]\}$$

is a harmonic function in the part of the exterior of C^* which lies in the lower half plane; and this function vanishes for

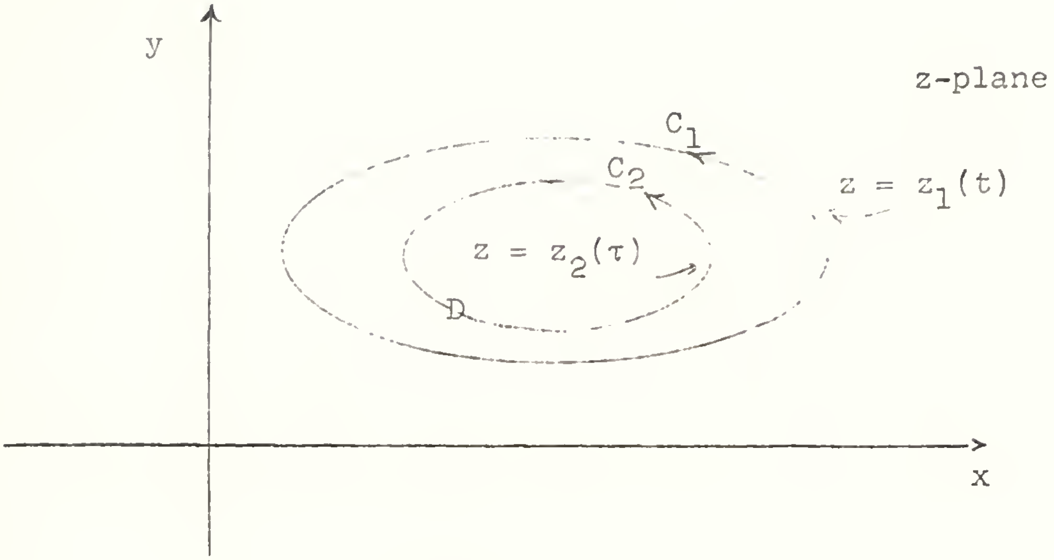

Fig. 1

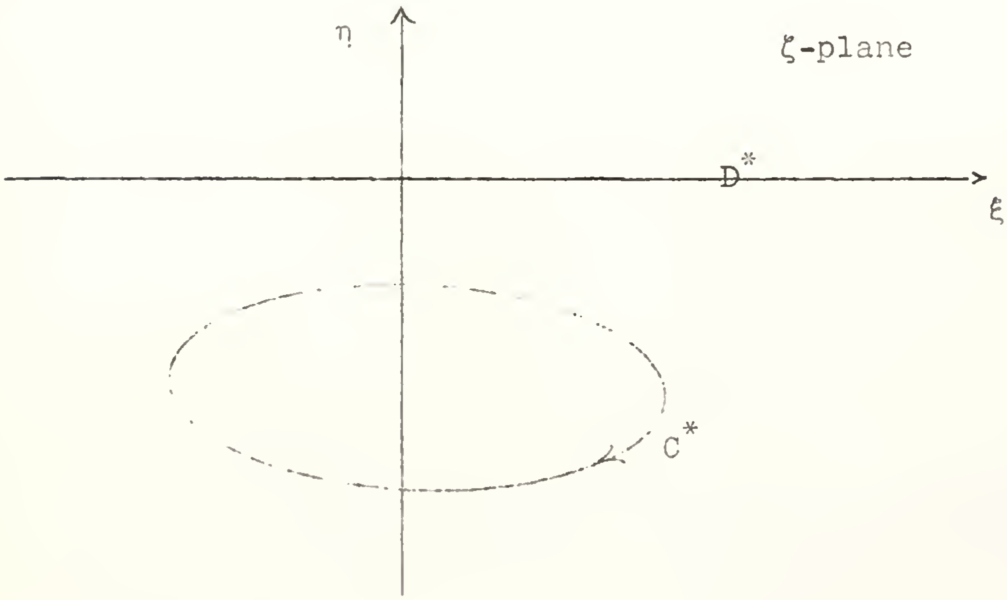

Fig. 2

$\operatorname{Im}\zeta = 0$. Hence if ζ^* is a point on C^*, harmonic continuation across $\operatorname{Im}\zeta = 0$ gives

$$\lim_{\zeta\to\zeta^*} \operatorname{Re}\{F[M(\zeta)] - H[M(\zeta)]\} = - \lim_{\zeta\to\zeta^*} \operatorname{Re}\{F[M(\bar{\zeta})]-H[M(\bar{\zeta})]\}$$

or

$$\lim_{\zeta\to\zeta^*}\operatorname{Re}\{F[M(\zeta)]\} = \lim_{\zeta\to\zeta^*}\operatorname{Re}\{H[M(\zeta)]+H[M(\bar{\zeta})]\} - \operatorname{Re}\{F[M(\bar{\zeta}^*)]\} .$$

For $\operatorname{Im}\zeta > 0$, $\operatorname{Re}\{F[M(\zeta)]\}$ is harmonic and its value at $\operatorname{Im}\zeta = 0$ is $\operatorname{Re}\{H[M(\xi)]\}$. Therefore

$$\operatorname{Re}\{F[M(\bar{\zeta}^*)]\} = \frac{1}{\pi} \operatorname{Im} \int_{-\infty}^{\infty} \frac{\operatorname{Re}\{H[M(\xi)]\}\, d\xi}{\xi - \bar{\zeta}^*} .$$

A return to the z-plane gives

$$(4.2) \qquad \begin{aligned} \phi(t) &= \lim_{z\to z_1(t)} \operatorname{Re}\{F(z)\} \\ &= \lim_{z\to z_1(t)} \operatorname{Re}\{H(z)+H(M[\overline{m(z)}])\} \\ &\quad - \frac{1}{\pi} \operatorname{Im} \int_c^d \frac{\operatorname{Re}\{H[z_2(\tau)]\}d\, m[z_2(\tau)]}{m[z_2(\tau)]-\overline{m[z_1(t)]}} \end{aligned}$$

or

$$(4.3) \qquad \begin{aligned} \phi(t) = &\lim_{z\to z_1(t)} \operatorname{Re}\{H(z)+H(M[\overline{m(z)}])\} \\ &- \frac{1}{\pi} \operatorname{Im} \int_c^d \operatorname{Re}\{H[z_2(\tau)]\} \frac{d}{d\tau} \ln\{m[z_2(\tau)]-\overline{m[z_1(t)]}\}\, d\tau \end{aligned}$$

for the solution of (4.1).

H. Bateman [3] considered the special case in which C_2 is given by $z = z_2(\tau) = \tau + ih$; C_1 is given by $z = z_1(t)$ with $\operatorname{Im} z_1(t) < h$; and D contains the point at infinity as shown in Fig. 3.

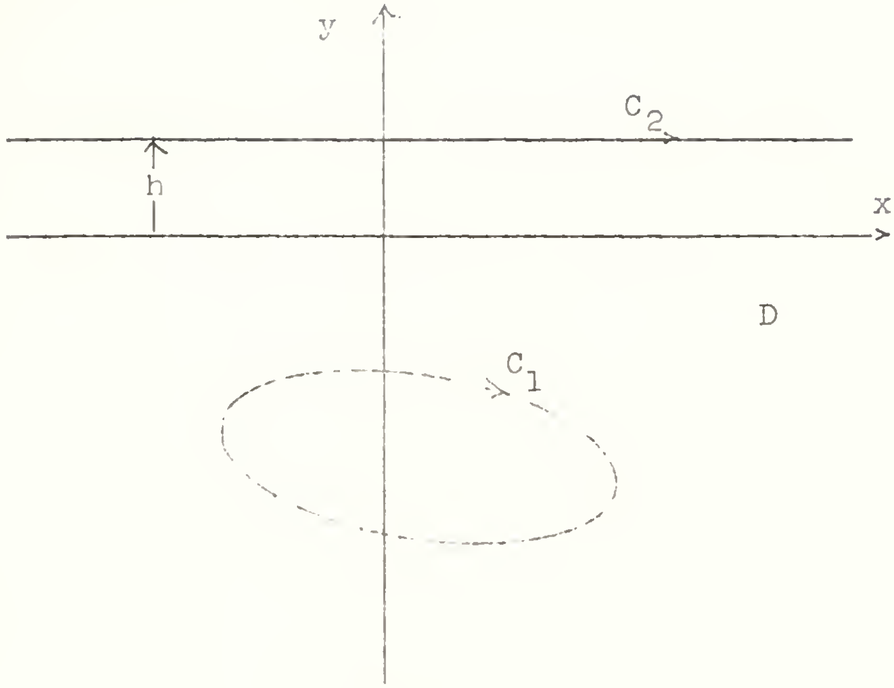

Fig. 3

Here the mapping function is

$$\zeta = m(z) = z - ih$$

$$z = M(\zeta) = \zeta + ih$$

and (4.2) shows that if

$$\psi(x,y) = \int_a^b \bar{G}(x,y;t)\ \phi(t)\ dt$$

is a regular potential function in D with boundary value $\phi(t)$ on C_1, then the solution of

$$f(\tau) = f[z_2(\tau)-ih] = \int_a^b \bar{G}(\tau,h;t)\ \phi(t)\ dt$$

is

$$\phi(t) = \lim_{z \to z_1(t)} \{f(z-ih)+f(\bar{z}+ih)\} - \frac{1}{\pi}\ \mathrm{Im} \int_{-\infty}^{\infty} \frac{f(\tau)\ d\tau}{\tau-x_1(t)+i[y_1(t)-h]}\ .$$

This agrees with Bateman's result.

The above results can be extended and specialized in various ways but we will consider only one more simple specialization which may be of interest. Suppose that a part Γ of the curve C_2 is brought into coincidence with a part of C_1, and that the remaining part of C_2 becomes a line segment L. For this case we can propose finding the value of $\psi(x,y)$ on $C_1-\Gamma$, where ψ is a potential function in D with its values prescribed on Γ and L. The method of this section can still be applied and the first step is to map the domain bounded by Γ+L into an upper half plane. By way of illustration let us find $\psi(x,0) = \phi(x)$ for $x > 0$ if $\psi(x,y)$ is known to satisfy

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \qquad y > 0$$

$$\psi(x,0) = 0 \qquad x < 0$$

$$\psi(r \cos \alpha,\ r \sin \alpha) = f(r) \qquad r > 0 \ ; \ \alpha < \frac{\pi}{2} \ .$$

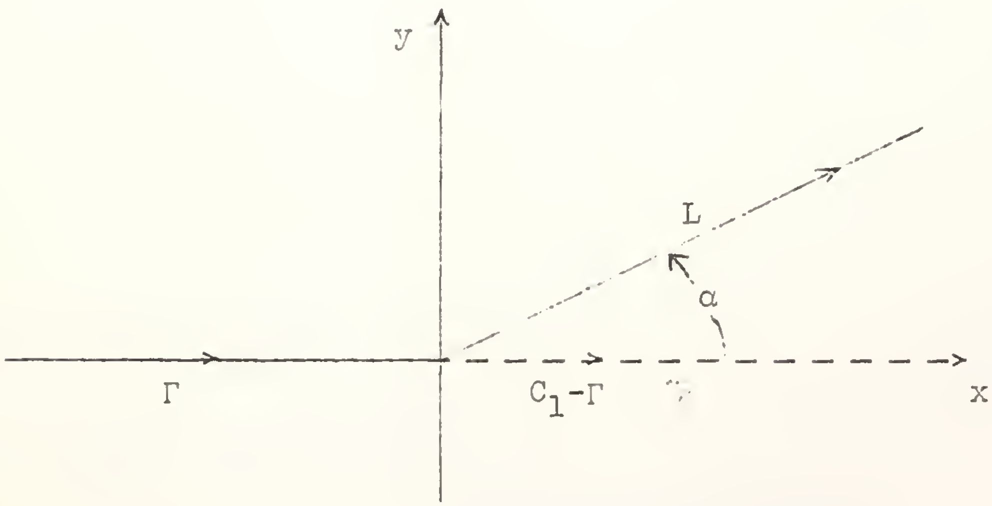

Fig. 4

The function $\psi(x,y)$ is given by

$$\psi(x,y) = \frac{1}{\pi}\int \frac{y\ \phi(\xi)\ d\xi}{(\xi-x)^2+y^2}$$

and if $z_2(\tau) = \tau e^{i\alpha}$, the problem is to solve

$$f(\tau) = f[e^{-i\alpha}z_2(\tau)] = \frac{1}{\pi}\int_0^\infty \frac{(\tau\sin\alpha)\ \phi(\xi)\ d\xi}{\xi^2+\tau^2-2\xi\tau\cos\alpha}$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\phi(\xi)\ d\xi}{\xi-\tau\ e^{i\alpha}}$$

for $\phi(\xi)$. The function $f(w)$ must be analytic for $w = \tau$ and as we can see from the integrals, $f(r)$ must allow analytic continuation to $f(w)$ where w is complex and $-\alpha < \arg w < \alpha$. The function

$$\zeta = [z\ e^{-i\alpha}]^{\frac{\pi}{\pi-\alpha}} = m(z)$$

$$z = e^{i\alpha}\ \zeta^{\frac{\pi-\alpha}{\pi}} = M(\zeta)$$

maps the section bounded by Γ+L into the upper half of a ζ-plane sheet and under this mapping the image of $z = \rho$, $\rho > 0$, is the ray $\zeta = \rho\exp\{-i\alpha\pi/\pi-\alpha\}$ in the lower half of the same sheet. With

$$z_2(\tau) = \tau\ e^{i\alpha}$$

$$z_1(t) = t$$

$$H(z) = f(z\ e^{-i\alpha})$$

the formula (4.2), that is,

$$\phi(t) = \lim_{z\to z_1(t)} \operatorname{Re}\{H(z)+H(M[m(z)])\}$$

$$-\frac{1}{\pi}\operatorname{Im}\int_c^d \frac{\operatorname{Re}\{H[z_2(\tau)]\}\ d\ m[z_2(\tau)]}{m[z_2(\tau)] - \overline{m[z_1(t)]}}$$

is applicable; and we find

$$\phi(t) = \lim_{\theta \to \alpha} [f(t\, e^{-i\theta}) + f(t\, e^{i\theta})] - \frac{1}{\pi-\alpha} \, \mathrm{Im} \int_0^\infty \frac{f(\tau)\, \tau^{\frac{\alpha}{\pi-\alpha}}\, d\tau}{\tau^{\frac{\pi}{\pi-\alpha}} - [t\, e^{i\alpha}]^{\frac{\pi}{\pi-\alpha}}} .$$

The reader will perhaps find it interesting to supply the modifications which are necessary when $\frac{\pi}{2} < \alpha < \pi$.

5. The Boundary Values of a Biharmonic Function which is Prescribed Along Two Interior Paths.

Problems, analogous to those studied in the previous sections, can also be proposed for the two-dimensional biharmonic equation which is a dominant equation in the theory of elasticity. We proceed to discuss one of these problems and thereby suggest some other applications of the inversion formulas found above. The problem we choose is this. Show how to find, in closed form, formulas for $\psi(r,\theta)$ and $\psi_\rho(r,\theta)$ if it is given that $\psi(\rho,\theta)$ satisfies

$$\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right\}\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2}\right\} = 0 \qquad \begin{array}{l} 0 \le \rho < r \\ 0 \le \theta \le 2\pi \end{array}$$

$$\psi(a,\theta) = f(\theta) \qquad 0 < a < b$$

$$\psi(b,\theta) = g(\theta) \qquad a < b < r .$$

If $\psi(\rho,\theta)$ is a biharmonic function in $0 \le \rho < r$ it is well known that

$$\psi(\rho,\theta) = \frac{(r^2-\rho^2)^2}{2\pi r} \int_0^{2\pi} \frac{[r-\rho\cos(\alpha-\theta)]\,\psi(r,\alpha)}{[r^2-2r\rho\cos(\alpha-\theta)+\rho^2]^2}\,d\alpha$$
$$+ \frac{(r^2-\rho^2)^2}{4\pi r} \int_0^{2\pi} \frac{\psi_\rho(r,\alpha)\,d\alpha}{r^2-24\rho\cos(\alpha-\theta)+\rho^2} .$$

Hence the problem just proposed is equivalent to solving the following pair of integral equations:

$$f(\theta) = \frac{(r^2-a^2)^2}{2\pi r} \int_0^{2\pi} \frac{[r-a\cos(\alpha-\theta)]\,\psi(r,\alpha)\,d\alpha}{[r^2-2ra\cos(\alpha-\theta)+a^2]^2} +$$
$$+ \frac{(r^2-a^2)^2}{4\pi r} \int_0^{2\pi} \frac{\psi_\rho(r,\alpha)\,d\alpha}{r^2-2ra\cos(\alpha-\theta)+a^2}$$

$$g(\theta) = \frac{(r^2-b^2)^2}{2\pi r} \int_0^{2\pi} \frac{[r-b\cos(\alpha-\theta)]\,\psi(r,\alpha)\,d\alpha}{[r^2-2rb\cos(\alpha-\theta)+b^2]^2} +$$
$$+ \frac{(r^2-b^2)^2}{4\pi r} \int_0^{2\pi} \frac{\psi_\rho(r,\alpha)\,d\alpha}{r^2-24b\cos(\alpha-\theta)+b^2} .$$

These equations could be solved by using finite transform techniques; but the course to the solution is easier and more satisfactory if we use the fact that

$$\psi(x,y) = \mathrm{Re}\,\{\bar{z}\,F_1(z) + G_1(z)\}$$

is a necessary and sufficient condition for $\psi(x,y)$ to be a biharmonic function in a domain D provided that $F_1(z)$ and $G(z)$ are analytic for $z = x + iy$ in D. The last equation can be rewritten as

$$\psi(x,y) = (r^2-z\bar{z})\,\mathrm{Re}\,F(z) + \mathrm{Re}\,G(z) ,$$

a form which is especially appropriate for the analysis of functions which are biharmonic in a disc with center at the

origin of the coordinate system. For our purposes it is most convenient if we take

$$\psi(\rho,\theta) = (r^2-\rho^2)\ \mathrm{Re}\ \frac{1}{2\pi i} \oint_C \frac{(\zeta+z)}{\zeta(\zeta-z)}\ \mu\left(\frac{1}{i}\ \ln \frac{\zeta}{r}\right) d\zeta \tag{5.1}$$
$$+ \mathrm{Re}\ \frac{1}{2\pi i} \oint_C \frac{(\zeta+z)}{\zeta(\zeta-z)}\ \nu\left(\frac{1}{i}\ \ln \frac{\zeta}{r}\right) d\zeta$$

where

$$\zeta = r\ e^{i\alpha} \quad ; \qquad z = \rho\ e^{i\theta}$$

and each of $\mu(\frac{1}{i}\ \ln \frac{\zeta}{r}) = \mu(\alpha)$; $\nu(\frac{1}{i}\ \ln \frac{\zeta}{r}) = \nu(\alpha)$ is real. From (5.1) we find

$$\psi_\rho(\rho,\theta) = -2\rho\ \mathrm{Re}\ \frac{1}{2\pi i} \cdot \oint_C \frac{(\zeta+z)}{\zeta(\zeta-z)}\ \mu\left(\frac{1}{i}\ \ln \frac{\zeta}{r}\right) d\zeta \tag{5.2}$$
$$- \frac{(r^2-\rho^2)}{\pi\rho} \frac{\partial}{\partial\theta}\ \mathrm{Re} \oint_C \frac{\mu(\frac{1}{i}\ \ln \frac{\zeta}{r})\ d\zeta}{\zeta-z}$$
$$- \frac{1}{\pi\rho} \frac{\partial}{\partial\theta}\ \mathrm{Re} \oint_C \frac{\nu(\frac{1}{i}\ \ln \frac{\zeta}{r})\ d\zeta}{\zeta - z}\ .$$

If we let $\rho \to r$ in (5.1) and (5.2) we obtain

$$\psi(r,\theta) = \nu(\theta)$$
$$\psi_\rho(r,\theta) = -\ 2r\ \mu(\theta) - \frac{1}{\pi r}\ \mathrm{Re} \oint_C \frac{\nu(\alpha)\ d\zeta}{\zeta-z}$$

which show how $\psi(r,\theta)$ and $\psi_\rho(r,\theta)$ depend on $\mu(\theta)$ and $\nu(\theta)$.

The problem has now been reduced to solving the system

$$f(\theta) = (r^2-a^2)\ \mathrm{Re}\ \frac{1}{2\pi i}\int_C \frac{(\zeta+ae^{i\theta})\ \mu(\alpha)\ d\zeta}{\zeta(\zeta-ae^{i\theta})} + \mathrm{Re}\ \frac{1}{2\pi i}\int_C \frac{\zeta + ae^{i\theta}}{\zeta(\zeta-ae^{i\theta})}\nu(\alpha)d\zeta$$

$$g(\theta) = (r^2-b^2)\ \mathrm{Re}\ \frac{1}{2\pi i}\int_C \frac{(\zeta+be^{i\theta})\mu(\alpha)\ d\zeta}{\zeta(\zeta-be^{i\theta})} + \mathrm{Re}\ \frac{1}{2\pi i}\int_C \frac{(\ +be^{i\theta})}{\zeta(\zeta-be^{i\theta})}\nu(\alpha)d\zeta$$

where $\zeta = r\ e^{i\alpha}$. These equations reduce to

$$f(\theta) = \frac{(r^2-a^2)}{2\pi}\int_0^{2\pi} \frac{[(r^2-a^2)\ \mu(\alpha)+\nu(\alpha)]}{r^2-24a\ \cos(\alpha-\theta)+a^2}\ d\alpha$$

$$g(\theta) = \frac{(r^2-b^2)}{2\pi}\int_0^{2\pi} \frac{[(r^2-b^2)\ \mu(\alpha)+\nu(\alpha)]}{r^2-24b\ \cos(\alpha-\theta)+a^2}\ d\alpha\ .$$

They can be solved by using the inversion formula given in Section 3. We find

$$(r^2-a^2)\mu(\theta)+\nu(\theta) = \lim_{\rho\to r-}\ [f(\theta+i\ \ln\tfrac{\rho}{a}) + f(\theta-i\ \ln\tfrac{\rho}{a})] - \frac{(r^2-a^2)}{2\pi}\int_0^{2\pi}\frac{f(\alpha)\ d\alpha}{r^2-2ra\ \cos(\alpha-\theta)+a^2}\ ;$$

$$(r^2-b^2)\mu(\theta)+\nu(\theta) = \lim_{\rho\to r-}\ [g(\theta)+i\ \ln\tfrac{\rho}{b}) + g(\theta-i\ \ln\tfrac{\rho}{b})] - \frac{(r^2-b^2)}{2\pi}\int\frac{g(\alpha)\ d\alpha}{r^2-2rb\ \cos(\alpha-\theta)+b^2}\ ;$$

and from these

$$(b^2-a^2)\nu(\theta) = \lim_{\rho\to r-} \left\{ \begin{array}{l} (r^2-a^2)\,[g(\theta+i\,\ln\frac{\rho}{b}) + g(\theta-i\,\ln\frac{\rho}{b})] \\ \\ -\,(r^2-b^2)[f(\theta+i\,\ln\frac{\rho}{a}) + f(\theta-i\,\ln\frac{\rho}{a})] \end{array} \right\}$$

$$+ \frac{(r^2-a^2)(r^2-b^2)}{2\pi} \int_0^{2\pi} \frac{f(\alpha)\,d\alpha}{r^2-24a\,\cos(\alpha-\theta)+a^2}$$

$$- \frac{(r^2-a^2)(r^2-b^2)}{2\pi} \int_0^{2\pi} \frac{g(\alpha)\,d\alpha}{r^2-24b\,\cos(\alpha-\theta)+b^2}$$

$$(b^2-a^2)\mu(\theta) = \lim_{\rho\to r-} \left\{ \begin{array}{l} f(\theta + i\,\ln\frac{\rho}{a}) + f(\theta - i\,\ln\frac{\rho}{a}) \\ \\ -g(\theta + i\,\ln\frac{\rho}{b}) - g(\theta - i\,\ln\frac{\rho}{b}) \end{array} \right\}$$

$$+ \frac{(r^2-b^2)}{2\pi} \int_0^{2\pi} \frac{g(\alpha)\,d\alpha}{r^2-2rb\,\cos(\alpha-\theta)+b^2}$$

$$- \frac{(r^2-a^2)}{2\pi} \int_0^{2\pi} \frac{f(\alpha)\,d\alpha}{r^2-2ra\,\cos(\alpha-\theta)+a^2}$$

6. The Inversion Formulas in Forms Suitable for Applications.

We have used several different methods to show

(a) if $\phi(t)$ is integralle over each finite interval

(b) if each of $\int_{-\infty}^{-c} \frac{\phi(t)\,dt}{t^2}$; $\int_{\infty}^{c} \frac{\phi(t)\,dt}{t^2}$ exists with $c > 0$

then the solution of

$$(6.1) \qquad f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\,\phi(t)\,dt}{(t-x)^2+a^2} \qquad \begin{array}{c} -\infty < x < \infty \\ a > 0 \end{array}$$

is

$$(6.2) \qquad \phi(x) = \lim_{y \to a-} [f(x+iy) + f(x-iy)] - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\,f(\xi)\,d\xi}{(t-x)^2+a^2}$$

for almost all values of x provided $f(z)$ is analytic for $-a < \operatorname{Im} z < a$. The "almost all" stipulation can be removed by contracting the class of functions to which ϕ may belong. Let us assume hereafter that instead of the condition (a) the function $\phi(t)$ is required to satisfy a uniform Hölder condition for all t; and that in addition the prescribed function is such that

$$\lim_{y \to a-} f(x+iy) = f(x+ia) \qquad \text{and} \qquad \lim_{y \to a-} f(x-iy) = f(x-ia) .$$

Under these conditions, and with condition (b), the inversion formula

$$(6.3) \qquad \phi(x) = f(x+ia) + f(x-ia) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\,f(\xi)\,d\xi}{(\xi-x)^2+a^2}$$

holds for all values of x.

With respect to the harmonic function

$$(6.4) \qquad \psi(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y\,\phi(t)\,dt}{(t-x)^2+y^2}$$

which is harmonic in the half-plane $y > 0$; and uniquely determined in $y > a$ by $\psi(x,a) = f(x)$; that is, by

$$(6.5) \qquad \psi(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(y-a)\, f(t)\, dt}{(t-x)^2+(y-a)^2} \qquad y > a ;$$

the inversion formula (6.3) is a continuation formula which allows us to continue $\psi(x,y)$ from the half-plane $-\infty < x < \infty$; $y \geq a$ into the strip $-\infty < x < \infty$; $0 < y \leq a$.

The inversion formula, or continuation formula (6.3) is quite satisfactory if $f(x)$ is analytically prescribed for $-\infty < x < \infty$; but if $f(x)$ is a tabular function derived from observations, the formula cannot be used without modification because $f(x+ia)$, for example, is not determinable from a finite number of real values of $f(x)$. In addition to this there is also an inherent difficulty. This second difficulty manifests itself when we realize that a small change in $f(x)$ may correspond to a large change in ϕ. For example, if $\phi(t) = \cos nt$ then

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a \cos nt\, dt}{(t-x)^2 + a^2} = e^{-an} \cos nx$$

and hence although the amplitude of variation of cos nt does not change with n, $f(x) = e^{-an} \cos nx$ may be made as small as we please by taking n sufficiently large. We therefore must admit that a small error in the measurement of $f(x)$ does not necessarily imply a small error in the computation of ϕ. This kind of instability, for integral equations of the first kind with difference kernels, was investigated by Kreisel [4] who met the difficulty by replacing the unknown function with a smoother function.

Our purpose here is to show how to derive some modifications of (6.3) under the assumption that errors in the data on $f(x)$ can be disregarded, or that rapidly changing solutions are ruled out. (If a smoothing process is necessary it can be applied after a modification of (6.3) has been obtained.) We do not propose here to enter an extensive numerical analysis or error analysis of any pertinent formula; our aim is rather to suggest elementary methods which may lead to practically useful formulas which can be substituted for (6.3). We also assume that $\underline{a}$, the distance from the datum line to the boundary, is known. If this is not the case, that is if $\psi(x,y)$ is actually harmonic in a domain which contains the half-plane $y > 0$; say the half-plane $y > -h$ ($h > 0$) , then (6.3) and its modifications simply give downward continuation formulas which enable us to establish the new datum line $y = 0$.

There are at least three ideas which can be used as guides to find approximations for (6.3):

(a) find a suitable approximation for $f(x+ia)+f(x-ia)=2 \operatorname{Re} f(x+ia)$;

(b) find an efficient formula which allows extrapolation from the known values $\psi(x,y)$, $y > a$, to the unknown $\phi(x) = \psi(x,0+)$;

(c) use or design iteration procedures which can be effectively applied to Poisson integral equations. These ideas are not independent. Each can in fact be expressed in terms of manipulations performed on the exponential e^{aw} which appears in the formula (2.10); however, it seems that it may be fruitful to categorize the ideas as we have. Although we are primarily interested in the third idea, a brief discussion of the other ideas is relevant.

In the first place, the sum $f(x+ia)+f(x-ia) = 2 \operatorname{Re} f(x+ia)$ in (6.3) can be replaced by its Taylor expansion so that we have

$$\phi(x) = 2 \operatorname{Re} \sum_{n=0}^{\infty} \frac{(ia)^n}{n!} f^n(x) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\ f(\xi)\ d\xi}{(t-x)^2+a^2}$$

(6.6) $$\phi(x) = 2 \sum_{m=0}^{\infty} \frac{(-1)^m a^{2m}}{(2m)!} f^{2m}(x) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\ f(\xi)\ d\xi}{(t-x)^2+a^2} .$$

This suggests that it may be feasible to introduce difference quotients. If we retain the first two terms of the Taylor expansion in (6.6), that is, take

(6.7) $$\phi(x) \cong 2\ f(x) - a^2\ f''(x) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\ f(\xi)\ d\xi}{(t-x)^2+a^2}$$

and replace $f''(x)$ by the central difference quotient

$$f''(x) \sim \frac{f(x+a) - 2f(x) + f(x-a)}{a^2}$$

we have

$$\phi(x) \cong 4\ f(x) - f(x+a) - f(x-a) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\ f(\xi)\ d\xi}{(\xi-x)^2+a^2}$$

which may be useful when a is small. In this formula we suppose that the integral is to be found by a process of numerical integration. The last formula can be written

(6.8) $$\phi(x) \cong 4\ f(x) - f(x+a) - f(x-a) - \psi(x,2a)$$

which is the simplest formula for the downward continuation of $\psi(x,y)$ when this function is known along $y = a$ and $y = 2a$. It represents, of course, no more than the substitution of the simplest partial difference equation for Laplace's equation.

It may be possible to approximate the tabulated $f(x)$ with a function $A(x)$, say a rational function approximation, such that $A(z)$ is analytic in $a < \text{Im } z < a$. If this is the case we can replace $f[z-i(a-\varepsilon)]$ in (2.6) by $A[z-i(a-\varepsilon)]$ and use

$$(6.9) \qquad \phi(x) \cong \lim_{\varepsilon \to 0} \text{Im} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{A[a-i(a-\varepsilon)]}{z-x-i\varepsilon} dz$$

to compute $\phi(x)$.

The derivatives of $f(x)$ in (6.6) can be replaced by derivatives of $\psi(x,y)$ with respect to y. We have $f(x) = \psi(x,a)$ and since $\psi(x,y)$ is harmonic for $y > 0$ it follows that $f''(x) = \psi_{xx}(x,a) = -\psi_{yy}(x,a)$. Using these facts we see that

$$(-1)^m f^{2m}(x) = \left. \frac{\partial^{2m} \psi(x,y)}{\partial y^{2m}} \right|_{y=a} .$$

For example, (6.7) can be written

$$(6.10) \qquad \phi(x) \cong 2f(x) + a^2 \psi_{yy}(x,a) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a\, f(\xi)\, d\xi}{(\xi-x)^2+a^2} .$$

If $a^2\psi_{yy}(x,a)$ is replaced by

$$a^2\psi_{yy}(x,a) \cong \psi(x,3a) - 2\psi(x,2a) + \psi(x,a)$$

then (6.10) becomes

$$(6.11) \qquad \phi(x) \cong \psi(x,3a) - 3\psi(x,2a) - 3\psi(x,a) .$$

If Δ is the difference operator defined by $\Delta\omega(x,y) = \omega(x,y+a)-\omega(x,y)$ the right hand side of (6.11) can be written as

$$\psi(x,a) - \Delta\psi(x,a) + \Delta^2\psi(x,a)$$

and this is the same as $F_2(0)$ where $F_2(y)$ is the second degree polynomial, $F_2(y) = \alpha_0 + \alpha_1 y + \alpha_2 y^2$, which satisfies

$$F_2(\kappa a) = \psi(x,\kappa a) \qquad \kappa = 1,2,3.$$

The formula

$$\phi(x) \cong \psi(x,a) - \Delta\psi(x,a) + \Delta^2\psi(x,a)$$

is then an extrapolation formula from known values $\psi(x,y)$, $y > a$, to $\phi(x)$. If we use a polynomial of the n-th degree, that is, $F_n(y) = \sum_{\kappa=0}^{n} \alpha_\kappa y^\kappa$ such that

$$F_n(\kappa a) = \psi(x,\kappa a) \qquad \kappa = 1,2,\ldots,n+1$$

we find

$$\phi(x) \cong \sum_{\kappa=0}^{n} (-1)^\kappa \Delta^\kappa \psi(x,a) . \tag{6.12}$$

If we admit the use of derivative values of $\psi(x,y)$ for $y > a$ it is clear that a large variety of other types of extrapolation formulae can be devised.

Let us turn now to an iteration procedure based on an analysis of Poisson's equation

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a\,\phi(t)\,dt}{(t-x)^2+a^2} \tag{6.13}$$

when it is written

$$P_a\,\phi = f .$$

Symbolically,

$$\phi(x) = \frac{1}{P_a}\,f = \frac{1}{1-(1-P_a)}\cdot f$$

$$\phi(x) = \sum_{\kappa=0}^{\infty} (1-P_a)^{\kappa} f .$$

Consider

(6.14) $$\phi_n(x) = \sum_{\kappa=0}^{n} (1-P_a)^{\kappa} f .$$

Since

$$(1-P_a)f = \psi(x,a) - \psi(x,2a) = -\Delta\psi(x,a)$$

the equation (6.14) is

$$\phi_n(x) = \sum_{\kappa=0}^{n} (-1)^{\kappa} \Delta^{\kappa} \psi(x,a)$$

which is the same as (6.12). We expect that if $\phi(x)$ is subject to certain restrictions then $\lim_{n\to\infty} \phi_n(x) = \phi(x)$. If $\phi_n(x)$ is known then $\phi_{n+1}(x)$ is given by

$$\phi_{n+1}(x) = \sum_{\kappa=0}^{n+1} (1-P_a)^{\kappa} f = \sum_{\kappa=-1}^{n} (1-P_a)^{\kappa+1} f$$

$$= f + \sum_{\kappa=0}^{n} (1-P_a)^{\kappa+1} f$$

$$= f + (1-P_a) \sum_{\kappa=0}^{n} (1-P_a)^{\kappa} f$$

or

(6.15) $$\phi_{n+1}(x) = f(x) + \phi_n(x) - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a\,\phi_n(t)\,dt}{(t-x)^2 + a^2} ;$$

a recurrence formula for successive ϕ_n's . The last equation also shows that ϕ_n satisfies the integral equation

(6.16) $$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a\,\phi_n(t)\,dt}{(t-x)^2 + a^2} = f(x) - [\phi_{n+1}(x) - \phi_n(x)]$$

$$= f(x) - (1-P_a)^{n+1} f .$$

If $\phi(t)$ satisfies (6.13) then (6.16) is the same as

$$P_a(\phi_n-\phi) = -(1-P_a)^{n+1} P_a \phi$$

which implies

$$\phi_n(x) - \phi(x) = -(1-P_a)^{n+1}\phi . \tag{6.17}$$

If we use the property $P_aP_b = P_{a+b}$, equation (6.17) becomes

$$\phi_n(x) - \phi(x) = -\{1-(n+1)P_a + \frac{(n+1)n}{n!} P_{2a} \ldots\}\phi$$

$$= -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\{\frac{\varepsilon}{(t-x)^2+\varepsilon^2} - \frac{(n+1)(a+\varepsilon)}{(t-x)^2+(a+\varepsilon)^2} \ldots\}\phi(t)\,dt$$

$$= -\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} e^{-\varepsilon\lambda}(1-e^{-a\lambda})^{n+1}\cos\lambda(t-x)d\lambda]\phi(t)dt.$$

It can be shown that $\lim_{n\to\infty}\phi_n(x) = \phi(x)$ if $\phi(x)$ satisfies one of several different conditions; but for simplicity we are now going to assume that $\phi(x)$ can be expressed as

$$\phi(x) = P_\delta\,\phi_1$$

where ϕ_1 is absolutely integrable. This implies that $\phi(z)$ is analytic in $-\delta < \operatorname{Im} z < \delta$. For this case (6.17) becomes

$$\phi_n(x) - \phi(x) = -(1-P_a)^{n+1} P_\delta\,\phi_1$$

or

$$\phi_n(x) - \phi(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty}[\int_0^{\infty} e^{-\delta\lambda}(1-e^{-a\lambda})^{n+1}\cos\lambda(t-x)d\lambda]\phi_1(t)\,dt.$$

Now

$$\int_0^\infty e^{-\delta\lambda}(1-e^{-a\lambda})^{n+1} \cos \lambda(t-x)\, d\lambda \le \int_0^N e^{-\delta\lambda}(1-e^{-a\lambda})^{n+1} d\lambda + \int_N^\infty e^{-\delta\lambda}\, d\lambda$$

$$\le (1-e^{-a\lambda_1})^{n+1} \frac{(1-e^{-\delta N})}{\delta} - \frac{e^{-\delta N}}{\delta}$$

where N is some fixed number independent of n; and $0 < \lambda_1 < N$. The number N can be chosen so large that $e^{-\delta N}/\delta$ is as small as we please, and then by choosing n sufficiently large $(1-e^{-a\lambda_1})^{n+1}$ can be made as small as we please since $0 < 1 - e^{-a\lambda_1} < 1$. It follows from this that $\phi_n(x)$ converges uniformly to $\phi(x)$ as $n \to \infty$.

From equation (6.17) we see that $\phi_n(x)$ can be written

$$\phi_n(x) = \{(n+1)P_a - \frac{(n+1)n}{2!} P_{2a} \dots (-1)^n P_{(n+1)a}\} \phi$$

or since $P_a\phi = f$

$$\phi_n(x) = (n+1)f + \frac{1}{\pi}\int_{-\infty}^{\infty} \{- \frac{(n+1)n}{2!} \cdot \frac{a^2}{(t-x)^2+a^2} \dots \frac{(-1)^n\, na}{(t-x)^2+(na)^2}\} f(t)\, dt$$

$$\phi_n(x) = (n+1)\, f(x) + \frac{1}{\pi}\int_{-\infty}^{\infty} f(t) \int_0^\infty \left[- \frac{(n+1)n}{2!} e^{-a\xi} + \frac{(n+1)n(n-1)}{3!} e^{-2a\xi} \dots (-1)^n e^{-na\xi} \right] \cos\xi(t-x)\, d\xi\, dt. \tag{6.18}$$

The inner integral of the last equation is equal to

$$I = \mathrm{Re} \int_0^\infty \left[- \frac{(n+1)n}{2!} e^{-2a\xi} + \frac{(n+1)n(n-1)}{3!} e^{-3a\xi} \dots (-1)^n e^{-(n+1)a\xi} \right] e^{[a+i(t-x)]\xi}\, d\xi$$

and two integrations by parts gives

$$I = \mathrm{Re}\,\{-\frac{1}{a+i(t-x)} - \frac{(n+1)na^2}{[a+i(t-x)]i(t-x)} \int_0^\infty e^{-a\xi}(1-e^{-a\xi})^{n-1}e^{i(t-x)\xi}d\xi\}.$$

Now

$$\int_0^\infty e^{-a\xi}(1-e^{-a\xi})^{n-1}e^{i(t-x)\xi}d\xi = \frac{1}{a}\int_0^1 z^{-i\beta}(1-z)^{n-1}\,dz$$

$$= \frac{1}{a}\;\frac{\Gamma(1-i\beta)\,\Gamma(n)}{\Gamma(n+1-i\beta)}$$

$$= \frac{1}{a}\;\frac{\Gamma(n)}{(n-i\beta)(n-1-i\beta)\dots(1-i\beta)}$$

where $\beta = (t-x)/a$. Hence

$$I = \mathrm{Re}\,\{-\frac{1}{a+i(t-x)} + \frac{(i)^n\,(n+1)!}{a(\beta-i)\beta(\beta+i)(\beta+2i)\dots(\beta+ni)}\}\;.$$

The substitution of this in (6.18) gives

$$\phi_n(x) = (n+1)f(x) - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{a\,f(t)\,dt}{(t-x)^2+a^2} + \frac{(n+1)!}{\pi}\,\mathrm{Re}(i)^n\int_{-\infty}^{\infty}\frac{f(x+a\beta)\,d\beta}{(\beta-i)\beta(\beta+i)\beta+2i)\dots(\beta+ni)}\;. \tag{6.19}$$

If we use the fact that

$$\frac{(n+1)!}{\pi}\,\mathrm{Re}(i)^n\int_{-\infty}^{\infty}\frac{d\beta}{(\beta-i)\beta(\beta+i)\dots(\beta+ni)} = -(n+1)+2$$

then (6.19) can be written in the numerically more useful form

$$\phi_n(x) = 2\,f(x) - \int_{-\infty}^{\infty}\frac{a\,f(t)\,dt}{(t-x)^2+a^2} + \frac{(n+1)!}{\pi}\,\mathrm{Re}(i)^n\int_{-\infty}^{\infty}\frac{[f(x+a\beta)-f(x)]\,d\beta}{(\beta-i)\beta(\beta+i)(\beta+2i)\dots(\beta+ni)} \tag{6.20}$$

This is an approximation which may be useful for numerical purposes if

$$\mathrm{Re}\ \frac{(i)^n}{(\beta-i)\beta(\beta+i)(\beta+2i)\ldots(\beta+ni)}$$

is tabulated for some values of n.

It is interesting to notice that the approximation (6.14), namely,

$$\phi_n(x) = \sum_{\kappa=0}^{n} (1-P_a)^{\kappa} f$$

can also be obtained by starting with the solution formula

$$\phi(x) = \frac{1}{\pi} \mathrm{Re} \int_0^{\infty} e^{aw} \int_{-\infty}^{\infty} e^{iw(t-x)} f(t)\, dt\, dw \tag{6.21}$$

derived in Section 2. The exponential e^{aw} can be expressed as

$$e^{aw} = \frac{1}{1-(1-e^{-aw})} = \sum_{\kappa=0}^{n} (1-e^{-aw})^{\kappa} + e^{aw}(1-e^{-aw})^{n+1}$$

and if we take

$$\phi_n(x) = \frac{1}{-} \mathrm{Re} \int_0^{\infty} \sum_{\kappa=0}^{n} (1-e^{-aw})^{\kappa} \int_{-\infty}^{\infty} e^{iw(t-x)} f(t)\, dt\, dw \tag{6.22}$$

we can show that (6.22) is the same as (6.14) if we grant that the order of integration can be changed.

The remarks of the last paragraph indicate that various iteration schemes, or numerical approximation formulas, can be developed after e^{aw} in (6.21) is replaced by some suitable approximation. For example it is interesting to examine the consequences of taking

$$e^{(a-\varepsilon)w} = e^{(y-\varepsilon)w}\, e^{-(y-a)w} = \sum_{n=0}^{\infty} \frac{[(y-\varepsilon)w]^n}{n!}\, e^{-(y-a)w}$$

$$= \sum_{n=0}^{\infty} \frac{[-(y-\varepsilon)]^n}{n!} \frac{\partial^n}{\partial y^n}\, e^{-(y-a)w} .$$

This suggests

$$\phi(x) = \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} \frac{[-(y-\varepsilon)]^n}{n!} \frac{\partial^n}{\partial y^n} \frac{1}{\pi} \operatorname{Re} \int_0^{\infty} e^{-(y-a)w} \int_{-\infty}^{\infty} e^{iw(t-x)} f(t)\, dt\, dw$$

or

$$(6.23) \quad \phi(x) = \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} \frac{[-(y-\varepsilon)]^n}{n!} \frac{\partial^n}{\partial y^n} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(y-a)\, f(t)\, dt}{(t-x)^2+(y-a)^2}$$

where it is to be understood that $y > a$. It is amusing to notice that (6.23) is an expression of the following.

$$\phi(x) = \lim_{\varepsilon \to 0} \psi(x,\varepsilon) = \lim_{\varepsilon \to 0} \psi[x, y-(y-\varepsilon)]$$

$$= \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} \frac{[-(y-\varepsilon)]^n}{n!} \frac{\partial^n}{\partial y^n} \psi(x,y) .$$

This proves (6.23) because if $y > a$

$$\psi(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(y-a)\, f(t)\, dt}{(t-x)^2+(y-a)^2} .$$

The formula (6.23) can be expressed in a better form. We have

$$(6.24) \quad \phi(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \operatorname{Im} \sum_{n=0}^{\infty} \frac{[-(y-\varepsilon)]^n}{n!} \frac{\partial^n}{\partial y^n} \int_{-\infty}^{\infty} \frac{f(t)\, dt}{t-x-i(y-a)}$$

$$(6.25) \quad \phi(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \operatorname{Im} \sum_{n=0}^{\infty} \frac{[-i(y-\varepsilon)]^n}{n!} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{\infty} \frac{f(t)\, dt}{t-x-i(y-a)} .$$

Expressed symbolically, this is

$$(6.26) \quad \phi(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \operatorname{Im} e^{-i(y-\varepsilon)\mathcal{D}} \int_{-\infty}^{\infty} \frac{f(t)\, dt}{t-x-i(y-a)}$$

where $\mathcal{D} \equiv \frac{\partial}{\partial x}$; and $y > a$. If we take $y = a+1$ then (6.26) becomes

$$(6.27) \quad \phi(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \operatorname{Im} e^{-i(a+1-\varepsilon)\mathcal{D}} \int_{-\infty}^{\infty} \frac{f(t)\, dt}{t-x-i} .$$

This means that after

$$F(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)\,dt}{t-x-i}$$

has been found then $\phi(x)$ is given by

$$\phi(x) = \lim_{\varepsilon\to 0} \operatorname{Im} F(x-ia-i+\varepsilon i) .$$

As far as the author knows, formula (6.27) is new. It appears to be the best of all of the formulas we have presented for the solution of

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a\,\phi(t)\,dt}{(t-x)^2+a^2} .$$

There is little if any room for an essential improvement of (6.27) as a formula to be used when $f(z)$ is prescribed analytically in $-a < \operatorname{Im} z < a$; and in addition to this, the formula seems to offer good possibilities for numerical analysis because it does not explicitly contain a term of the type $f(z+i\sigma)$, as the previous formulas, which involve one integration, do. Using observed values of $f(x)$ we can apply a process of numerical integration to

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)\,dt}{t-x-i}$$

and thus obtain $F_\alpha(x)$ as an analytic approximation to $F(x)$. The values of $\phi(x)$ can then be probed at various depths a below the datum line by using

$$\phi(x) = \lim_{\varepsilon\to 0} F_\alpha(x-ia-i+i\varepsilon) .$$

It should be noted that (6.23) can be expanded into

$$\phi(x) = \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} \frac{[-(y-\varepsilon)]^{2n}}{(2n)!} \frac{\partial^{2n}}{\partial y^{2n}} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(y-a)\ f(t)\ dt}{(x-t)^2+(y-a)^2}$$

$$+ \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} \frac{[-(y-\varepsilon)]^{2n+1}}{(2n+1)!} \frac{\partial^{2n+1}}{\partial y^{2n+1}} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(y-a)\ f(t)\ dt}{(x-t)^2+(y-a)^2}$$

or, what is the same thing,

$$(6.28) \quad \phi(x) = \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} \frac{(-1)^n (y-\varepsilon)^{2n}}{(2n)!} \frac{\partial^{2n}}{\partial x^{2n}} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(y-a)\ f(t)\ dt}{(x-t)^2+(y-a)^2}$$

$$+ \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} \frac{(-1)^n (y-\varepsilon)^{2n+1}}{(2n+1)!} \frac{\partial^{2n+1}}{\partial x^{2n+1}} \cdot \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-t)\ f(t)\ dt}{(x-t)^2+(y-a)^2} .$$

In operational form (6.28) is

$$(6.29) \quad \phi(x) = \lim_{\varepsilon \to 0} \cos(y-\varepsilon)\mathcal{D} \cdot \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(y-a)\ f(t)\ dt}{(t-x)^2+(y-a)^2}$$

$$- \lim_{\varepsilon \to 0} \sin(y-\varepsilon)\mathcal{D} \cdot \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(t-x)\ f(t)\ dt}{(t-x)^2+(y-a)^2} .$$

A recognition of these expansions is helpful when we seek the three-dimensional analog of (6.27).

7. Poisson's Integral Equation in Two Dimensions.

Let us turn now to a three-dimensional potential problem of the type discussed in Section 2. If $\psi(x,y,z)$ is a regular potential function in the half-space $z > 0$; and if $\psi(x,y,0+)=\phi(x,y)$ where $\phi(x,y)$ is continuous everywhere, then we know that $\psi(x,y,z)$ is uniquely given by

$$\psi(x,y,z) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{z\ \phi(\xi,\eta)\ d\xi\ d\eta}{[(\xi,x)^2+(\eta-y)^2+z^2]^{3/2}} . \tag{7.1}$$

If $\mu(x,y,a) = f(x,y)$ is a prescribed analytic function then $\phi(\xi,\eta)$ must satisfy

$$f(x,y) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{a\ \phi(\xi,\eta)\ d\xi\ d\eta}{[(\xi-x)^2+(\eta-y)^2+a^2]^{3/2}} \qquad \begin{array}{c} -\infty < x < \infty \\ -\infty < y < \infty \\ a > 0 \end{array} \tag{7.2}$$

which is the counterpart of Poisson's one-dimensional equation (1.1). As was pointed out in the introduction, equation (7.2) is of considerable interest since it arises in connection with certain kinds of geological prospecting. We proceed to give a brief analysis of (7.2) in order to substantiate the fact that the previous sections contain many results which continue to hold for the two and higher dimensional generalizations of the equations discussed in those sections.

The equation (7.2) can be solved in several different ways. The method which follows is based on the semi-group property of the operator $\mathcal{O}_a$. We define $\mathcal{O}_a\phi$ by

$$\mathcal{O}_a\ \phi = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{a\ \phi(\xi,\eta)\ d\xi\ d\eta}{[(\xi-x)^2+(\eta-y)^2+a^2]^{3/2}}$$

so that

$$\mathcal{O}_b\mathcal{O}_a\phi \equiv \frac{1}{4\pi^2}\iint_{-\infty}^{\infty}\frac{b}{[(\xi_1-x)^2+(\eta_1-y)^2+b^2]^{3/2}}\cdot\iint_{-\infty}^{\infty}\frac{a\ \phi(\xi,\eta)\ d\xi\ d\eta\ d\xi_1\ d\eta_1}{[(\xi-\xi_1)^2+(\eta-\eta_1)^2+a^2]^{3/2}}$$

$$= \omega(x,y,b)$$

where

$$\omega(x,y,z) = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{z}{[(\xi_1-x)^2+(\eta_1-y)^2+z^2]^{3/2}}\cdot\frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{a\ \phi(\xi,\eta)d\xi\ d\eta\ d\xi_1\ d\eta_1}{[(\xi-\xi_1)^2+(\eta-\eta_1)^2+a^2]^{3/2}}.$$

The function $\omega(x,y,z)$ is a regular potential function for $z > 0$ and

$$\omega(x,y,0+) = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{a\ \phi(\xi,\eta)\ d\xi\ d\eta}{[(\xi-x)^2+(\eta-y)^2+a^2]^{3/2}} ;$$

but we also have

$$\omega(x,y,0+) = \frac{1}{2\pi}\lim_{z\to 0}\iint_{-\infty}^{\infty}\frac{(z+a)\ \phi(\xi,\eta)\ d\xi\ d\eta}{[(\xi-x)^2+(\eta-y)^2+(z+a)^2]^{3/2}} ;$$

and since

$$\frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{(z+a)\ \phi(\xi,\eta)\ d\xi\ d\eta}{[(\xi-x)^2+(\eta-y)^2+(z+a)^2]^{3/2}}$$

is a potential function in $z > 0$ with the same boundary value as $\omega(x,y,z)$ we conclude from the uniqueness theorem that

$$\omega(x,y,z) = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{(z+a)\ \phi(\xi,\eta)\ d\xi\ d\eta}{[(\xi-x)^2+(\eta-y)^2+(z+a)^2]^{3/2}} .$$

Therefore

$$\mathcal{O}_b\mathcal{O}_a\ \phi = \omega(x,y,b) = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\frac{(b+a)\ \phi(\xi,\eta)\ d\xi\ d\eta}{[(\xi-x)^2+(\eta-y)^2+(b+a)^2]^{3/2}}$$

or

(7.3) $$\mathcal{O}_b\mathcal{O}_a\ \phi = \mathcal{O}_{a+b}\ \phi .$$

The property (7.3) can be used to solve (7.2). With $0 < z < a$, the application of $\mathcal{D}_z$ to each side of (7.2) produces

$$\mathcal{D}_z f = \frac{1}{2\pi}\iint_{-\infty}^{\infty} \frac{(z+a)\,\phi(\xi,\eta)\,d\xi\,d\eta}{[(\xi-x)^2+(\eta-y)^2+(z+a)^2]^{3/2}}$$

$$= \frac{1}{2\pi}\iint_{-\infty}^{\infty}\int_0^{\infty} r\,e^{-(a+z)r}\,J_0(r\rho)\,dr\,\phi(\xi,\eta)\,d\xi\,d\eta$$

where $\rho = \sqrt{(\xi-x)^2+(\eta-y)^2}$. The last equations is the same as

$$\mathcal{D}_z f = \frac{1}{2\pi}\iint_{-\infty}^{\infty}\int_0^{\infty} r[e^{-(a+z)r}+e^{-(a-z)r}]\,J_0(r\rho)\,dr\,\phi(\xi,\eta)\,d\xi\,d\eta$$

$$- \frac{1}{2\pi}\iint_{-\infty}^{\infty} \frac{(a-z)\,\phi(\xi,\eta)\,d\xi\,d\eta}{[(\xi-x)^2+(\eta-y)^2+(a-z)^2]^{3/2}} .$$

If we let $z \to a-0$ we find

$$\phi(x,y) = \lim_{z\to a-} \frac{1}{2\pi}\iint_{-\infty}^{\infty}\int_0^{\infty} r\,e^{-ar}\cdot 2\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}r^{2n}J_0(r\rho)dr\,\phi(\xi,\eta)d\xi\,d\eta$$

$$-\mathcal{D}_a f .$$

Since

$$\nabla^2 J_0(r\rho) \equiv \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)J_0(r\rho) = \left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right]J_0(r\rho)$$

$$= -\,r^2\,J_0(r\rho)$$

it follows that

$$\nabla^{2n}\,J_0(r\rho) = (-1)^n\,r^{2n}\,J_0(r\rho)$$

and hence

$$\phi(x,y) = \lim_{z\to a-} 2\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}\,\nabla^{2n}\cdot\frac{1}{2\pi}\iint_{-\infty}^{\infty}\int_0^{\infty} re^{-ar}J_0(r\rho)\,dr\,\phi(\xi,\eta)d\xi\,d\eta$$

$$-\mathcal{D}_a f$$

$$\phi(x,y) = \lim_{z\to a-} 2\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}\nabla^{2n}}{(2n)!} \mathcal{O}_a \phi - \mathcal{O}_a f$$

which yields

$$(7.4) \qquad \phi(x,y) = \lim_{z\to a-} 2\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \nabla^{2n} f(x,y)$$
$$- \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{a\, f(\xi,\eta)\, d\xi\, d\eta}{[(\xi-x)^2+(\eta-y)^2+a^2]^{3/2}}$$

for the solution of (7.2).

We can now ask for verions of (7.4) which can be used to compute $\phi(x,y)$ when $f(x,y)$ is a function derived from experimental observations. For numerical purposes there are many approximations that can be used in place of (7.4) and these approximations can be deduced by using the methods of Section 6. In the main part of what follows we will confine ourselves to the derivation of a formula which can be based on an iteration procedure analogous to that presented in the last section.

The solution of (7.2) can be expressed in the symbolic form

$$\phi = \frac{1}{\mathcal{O}_a} f = \frac{1}{1-(1-\mathcal{O}_a)} f$$

and from this we find the formal symbolic solution

$$\phi = \sum_{\kappa=0}^{\infty} (1-\mathcal{O}_a)^{\kappa} f .$$

If we suppose that

$$(7.5) \qquad \phi_n = \sum_{\kappa=0}^{n} (1-\mathcal{O}_a)^{\kappa} f$$

is known then ϕ_{n+1} is given by the recurrence formula

$$\phi_{n+1}(x) = f(x) + \phi_n(x) - \frac{1}{2} \iint_{-\infty}^{\infty} \frac{a\, \phi_n(\xi,\eta)\, d\xi\, d\eta}{[(\xi-x)^2+(\eta-y)^2+a^2]^{3/2}} .$$

This also means that ϕ_n must satisfy

$$(7.6)\qquad \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} \frac{a\ \phi_n(\xi,\eta)\ d\xi\ d\eta}{[(\xi-x)^2+(\eta-y)^2+a^2]^{3/2}} = f - (\phi_{n+1}-\phi_n)$$

$$= f - (1-\mathcal{O}_a)^{n+1}\ f\ .$$

As in Section 6 it can be shown that if, for example, $\phi = \mathcal{O}_\delta\phi_1$ with ϕ_1 absolutely integrable, then ϕ_n converges uniformly to ϕ as $n\to\infty$.

Use of the semi-group property of $\mathcal{O}_a$ allows us to write (7.6) as

$$\mathcal{O}_a\ \phi_n = \{(n+1)\,\mathcal{O}_a - \frac{(n+1)n}{2!}\,\mathcal{O}_{2a}\ \ldots\ (-1)^n\,\mathcal{O}_{(n+1)a}\}\ f$$

from which

$$(7.7)\qquad \phi_n(x,y) = (n+1)f(x,y) - \frac{(n+1)n}{2!}\,\mathcal{O}_a\ f\ \ldots\ (-1)^n\,\mathcal{O}_{na}\ f\ .$$

If we use polar coordinates

$$\xi - x = \rho\cos\theta \qquad\qquad \eta - y = \rho\sin\theta$$

$$f(\xi,\eta) = f(x+\rho\cos\theta,\ y+\rho\sin\theta) = F(\rho,\theta;x,y)$$

then (7.7) becomes

$$(7.8)\qquad \phi_n(x,y) = (n+1)\ f(x,y)$$

$$+\frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi}\left\{\begin{array}{c} -\dfrac{(n+1)n}{2!}\ \dfrac{a}{(\rho^2+a^2)^{3/2}} + \dfrac{(n+1)n(n-1)}{3!}\cdot\dfrac{2a}{(\rho^2+4a^2)^{3/2}} \\ \ldots \\ \ldots\ \dfrac{(-1)^n\ na}{(\rho^2+n^2a^2)^{3/2}} \end{array}\right\}\rho F d\theta\, d\rho\ .$$

This can be expressed in essentially the same form as its one-dimensional counterpart if we notice that

$$\frac{\kappa a}{[\rho^2+\kappa^2 a^2]^{3/2}} = \int_0^\infty r\, e^{-\kappa ar}\, J_0(r\rho)\, dr$$

$$J_0(r\rho) = \frac{2}{\pi}\int_\rho^\infty \frac{\sin r\tau\, d\tau}{\sqrt{\tau^2-\rho^2}} = \frac{2}{\pi}\,\mathrm{Im}\int_\rho^\infty \frac{e^{ir\tau}\, dt}{\sqrt{\tau^2-\rho^2}}$$

$$\frac{\kappa a}{[\rho^2+\kappa^2 a^2]^{3/2}} = -\frac{1}{2\pi}\mathrm{Re}\int_\rho^\infty \frac{1}{\sqrt{\tau^2-\rho^2}}\,\frac{\partial}{\partial\tau}\int_0^\infty e^{-\kappa ar} e^{ir\tau}\, dr\, d\tau\ .$$

Hence

$$\phi_n(x,y) = (n+1)\, f(x,y)$$

$$-\frac{1}{\pi^2}\int_0^\infty\int_0^{2\pi}\rho\, F\, d\theta\, d\rho \cdot \mathrm{Re}\int_\rho^\infty \frac{1}{\sqrt{\tau^2-\rho^2}}\,\frac{\partial}{\partial\tau}\int_0^{\infty\infty} e^{ir\tau}\{-\frac{(n+1)n}{2!}\, e^{-ar}\ldots\}\, dr\, d\tau\ .$$

The integral

$$\mathrm{Im} = \mathrm{Re}\int_0 e^{ir\tau}\left\{\begin{array}{c} -\frac{(n+1)n}{2!}\, e^{-ar} + \frac{(n+1)n(n-1)}{3!}\, e^{-2ar} \\ \vdots \\ \ldots\ (-1)^n\, e^{-nar}\end{array}\right\} dr$$

was evaluated in Section 6 and it is equal to

$$\mathrm{Im} = \mathrm{Re}\,\{-\frac{1}{a+i\tau} + \frac{(i)^n\,(n+1)!}{a(\beta-i)\beta(\beta+i)\ldots(\beta+ni)}\}$$

where $\beta = \tau/a$. We therefore have

$$(7.9)\quad \phi_n(x,y) = (n+1)\, f(x,y) + \frac{1}{\pi^2}\int_0^\infty\int_0^{2\pi}\rho\, F\, d\theta\, d\rho\int_\rho \frac{1}{\sqrt{\tau^2-\rho^2}}\,\frac{\partial}{\partial\tau}\left(\frac{a}{a^2+\tau^2}\right) d\tau$$

$$-\frac{(n+1)!}{a\pi^2}\int_0^\infty\int_0^{2\pi}\rho\, F\, d\theta\, d\rho\ \mathrm{Re}\int_\rho^\infty \frac{1}{\sqrt{\tau^2-\rho^2}}\,\frac{\partial}{\partial\tau}\mathcal{F}(\beta)\, d\tau$$

where

$$\mathcal{F}(\beta) = \frac{(i)^n}{(\beta-i)\beta(\beta+i)(\beta+2i)\dots(\beta+ni)} \quad .$$

The form of equation (7.9) can be improved by noting that

$$\int^{\infty} \frac{1}{\sqrt{\tau^2-\rho^2}} \frac{\partial}{\partial\tau} \left(\frac{a^2}{a^2+\tau^2}\right) d\tau = - \frac{\pi}{2} \frac{a}{[\rho^2+a^2]^{3/2}}$$

and that

$$- \frac{(n+1)!}{a\pi^2} \int_0^{\infty}\int_0^{2\pi} \rho \; d\theta \; d\rho \cdot \mathrm{Re} \int_{\rho}^{\infty} \frac{1}{\sqrt{\tau^2-\rho^2}} \frac{\partial}{\partial\tau} \mathcal{F}(\beta) \; d\tau$$

$$= - (n+1) + 2 \; .$$

The use of these results allows us to write

$$(7.10) \quad \phi_n(x,y) = 2 \; f(x,y) - \frac{1}{2\pi} \int_0^{\infty}\int_0^{2\pi} \frac{a \; \rho \; F \; d\theta \; d}{[\rho^2+a^2]^{3/2}}$$

$$- \frac{(n+1)!}{a\pi^2} \int_0^{\infty}\int_0^{2\pi} [F-f(x,y)]\rho \; d\theta \; d\rho \; \mathrm{Re} \int_{\rho}^{\infty} \frac{1}{\sqrt{\tau^2-\rho^2}} \frac{\partial}{\partial\tau} \mathcal{F}(\beta) \; d\tau$$

or

$$(7.11) \quad \phi_n(x,y) = 2 \; f(x,y) - \frac{1}{2\pi} \int_0^{2\pi}\int_0^{\infty} \frac{a \; f(x+\rho \cos \theta, y+\rho \; \sin \theta)\rho \; d\rho \; d\theta}{[\rho^2+a^2]^{3/2}}$$

$$- \frac{(n+1)!}{\pi^2} \int_0^{2\pi}\int_0^{\infty} [f(x+a\rho \cos \theta, y+a\rho \sin \theta)-f(x,y)]\rho \; d\rho \; d\theta \cdot$$

$$\cdot \; \mathrm{Re} \int_{\rho}^{\infty} \frac{1}{\sqrt{\beta^2-\rho^2}} \frac{\partial}{\partial\beta} \mathcal{F}(\beta) \; d\beta$$

which may be useful for numerical analysis if

$$\mathrm{Re} \int_{\rho}^{\infty} \frac{1}{\sqrt{\beta^2-\rho^2}} \frac{\partial}{\partial\beta} \mathcal{F}(\beta) \; d\beta$$

is tabulated.

Let us conclude with a derivation of an analog of (6.27). The unknown function $\phi(x,y)$ can be expressed as

$$\phi(x,y) = \lim_{\varepsilon\to 0} \psi(x,y,\varepsilon) = \lim_{\varepsilon\to 0} \psi[x,y,z-(z-\varepsilon)]$$

$$= \lim_{\varepsilon\to 0} \sum_{n=0}^{\infty} \frac{[-(z-\varepsilon)]^n}{n!} \frac{\partial^n}{\partial z^n} \psi(x,y,z)$$

which implies

$$\phi(x,y) = \lim_{\varepsilon\to 0} \sum_{n=0}^{\infty} \frac{[-(z-\varepsilon)]^n}{n!} \frac{\partial^n}{\partial z^n} \cdot \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{(z-a)\, f(\xi,\eta)\, d\xi\, d\eta}{[(\xi-x)^2+(\eta-y)^2+(z-a)^2]^{3/2}}$$

$$= \lim_{\varepsilon\to 0} e^{-(z-\varepsilon)\mathcal{D}} \cdot \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{(z-a)\, f(\xi,\eta)\, d\xi\, d\eta}{[(\xi-x)^2+(\eta-y)^2+(z-a)^2]^{3/2}}$$

where $\mathcal{D} \equiv \frac{\partial}{\partial z}$ and $z > a$. This can be expanded into

$$\phi(x,y) = \lim_{\varepsilon\to 0} \sum_{n=0}^{\infty} \frac{(z-\varepsilon)^{2n}}{(2n)!} \frac{\partial^{2n}}{\partial z^{2n}} \cdot \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{(z-a)\, f(\xi,\eta)\, d\xi\, d\eta}{[(\xi-x)^2+(\eta-y)^2+(z-a)^2]^{3/2}}$$

$$+ \lim_{\varepsilon\to 0} \sum_{n=0}^{\infty} \frac{(z-\varepsilon)^{2n+1}}{(2n+1)!} \frac{\partial^{2n+2}}{\partial z^{2n+2}} \cdot \frac{1}{\pi} \iint_{-\infty}^{\infty} \frac{f(\xi,\eta)\, d\xi\, d\eta}{[(\xi-x)^2+(\eta-y)^2+(z-a)^2]^{1/2}}$$

$$= \lim_{\varepsilon\to 0} \sum_{n=0}^{\infty} \frac{(-1)^n (z-\varepsilon)^{2n}}{(2n)!} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^n \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{(z-a)\, f(\xi,\eta)\, d\xi\, d\eta}{[(\xi-x)^2+(\eta-y)^2+(z-a)^2]^{3/2}}$$

$$+ \lim_{\varepsilon\to 0} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (z-\varepsilon)^{2n+1}}{(2n+1)!} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^{n+1} \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{f(\xi,\eta)\, d\xi\, d\eta}{[(\xi-x)^2+(\eta-y)^2+(z-a)]^{1/2}} .$$

In particular if we take $z = a+1$

$$\phi(x,y) = \lim_{\varepsilon\to 0} \sum_{n=0}^{\infty} \frac{(-1)^n (a+1-\varepsilon)^{2n}}{(2n)!} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^n \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{f(\xi,\eta)\, d\xi\, d\eta}{[(\xi-x)^2+(\eta-y)^2+1]^{3/2}}$$

$$+ \lim_{\varepsilon\to 0} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (a+1-\varepsilon)^{2n+1}}{(2n+1)!} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^{n+1} \frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{f(\xi,\eta)\, d\xi\, d\eta}{[(\xi-x)^2+(\eta-y)^2+1]^{1/2}} .$$

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